Exact multilocal renormalization of the effective action: Application to the random sine Gordon model statics and nonequilibrium dynamics

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We extend the exact multilocal renormalization group (RG) method to study the flow of the effective action functional. This important physical quantity satisfies an exact RG equation which is then expanded in multilocal components. Integrating the nonlocal parts yields a closed exact RG equation for the local part, to a given order in the local part. The method is illustrated on the $O(N)$ model by straightforwardly recovering the η exponent and scaling functions. Then it is applied to study the glass phase of the Cardy-Ostlund, random phase sine Gordon model near the glass transition temperature. The static correlations and equilibrium dynamical exponent z are recovered and several results are obtained, such as the equilibrium two-point scaling functions. The nonequilibrium, finite momentum, two-time t,t' response and correlations are computed. They are shown to exhibit scaling forms, characterized by exponents $\lambda_R \neq \lambda_C$, as well as universal scaling functions that we compute. The fluctuation dissipation ratio is found to be nontrivial and of the form $X[q^z(t-t'),t/t']$. Analogies and differences with pure critical models are discussed.

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I. INTRODUCTION

Recently a method was devised, the exact multilocal renormalization group (EMRG) [1], to obtain perturbative renormalization group equations from first principles, in a controlled way to any order, and for an arbitrary smooth cutoff function. It starts, as numerous previous exact RG studies [2–12], from the exact Polchinski-Wilson renormalization group equation [13,14] for the action functional $\mathcal{S}(\phi)$. The next step, however, consists in splitting it onto local and higher multilocal components [15], and integrating exactly all multilocal components in terms of the local part. This yields an exact and very general RG flow equation for the local part of the action, i.e., a function, expressed in an expansion in powers of the local part.

The aim of this paper is first to develop a similar method using instead the effective action functional $\Gamma(\phi)$. This is needed because $\Gamma(\phi)$ is a very important physical object, both as the generating function of proper vertices, and related to the probability distribution of an arbitrary macroscopic mode ϕ_q [16]. A multilocal expansion is also performed and yields a RG equation again in terms of the local part. The major advantage compared to the previous method [1] is that one actually follows directly physical observables and that correlations are immediately obtained (while in the previous method one had to use a second formula to compute correlations from the flowing action). The price to pay is a slightly more involved RG equation, but this inconvenience arises only at higher orders. As a simple check, the η exponent of the $O(N)$ model will be recovered to lowest order.

A motivation to develop such EMRG methods comes from disordered models. The physics of these being more complex than standard field theories for pure systems, it is useful to be able to control the RG procedure. This is crucial, for instance, in the functional RG (FRG) which describes pinned elastic manifolds [17–19], relevant for, e.g., superconductors and density waves [20–24], and the EMRG has been applied to study that problem [1,15]. Here, and this is the second aim of this paper, we will study another instance of a glass phase, arising in the random phase sine Gordon model excluding vortices, as discovered by Cardy and Ostlund [25]. This model has been studied extensively, in its statics [19,26–31] and its dynamics [32,33], as one of the simplest but nontrivial examples of a topologically ordered glass, a continuation to two dimensions [21,34] of the fixed point describing the Bragg glass phase in three dimensions [19]. We first show that the present method allows to recover very simply and in a controlled way previous results for correlation functions in the statics and in the equilibrium dynamics. Next we obtain results, such as the full scaling functions for both equilibrium and nonequilibrium dynamics. We obtain the corresponding exponents λ and θ. We also obtain the full and nontrivial behavior of the fluctuation dissipation ratio in the glass phase.

The outline of the paper is as follows. First in Sec. II we derive the EMRG method for the effective action, and give the explicit general lowest order RG equations. In Sec. III we apply these RG equations to the pure $O(N)$ model, as a test of the method. In Sec. IV we consider the Cardy-Ostlund (CO) model statics. In Sec. V we study the CO model equilibrium dynamics. Section VI is devoted to the nonequilibrium dynamics of the CO model. All calculational details are contained in the appendixes.

II. METHOD

A. Exact RG method

We want to study interacting bosonic degrees of freedom described by a set of fields denoted $\phi \equiv \phi_x^i$ where x is the position in space and i a general label denoting any quantity which will not undergo the coarse graining (e.g., field indices, spin, replica indices, and additional coordinate). The problem is defined by an action functional,

$$\mathcal{S}(\phi)=\frac{1}{2}\phi:G^{-1}:\phi+\mathcal{V}(\phi) \tag{1}$$

and by the functional integral (i.e., the partition function) $Z=\int D\phi e^{-\mathcal{S}(\phi)}$. The action consists of a quadratic part ($G_{ij}^{xy}=G_{ji}^{yx}$ is a symmetric invertible matrix) and $\mathcal{V}(\phi)$ the interaction, a functional of ϕ. The notation ":" denotes full contractions over x,i [i.e., $\phi:G^{-1}:\phi=\Sigma_{ij}\int_{xy}\phi_x^i(G^{-1})_{ij}^{xy}\phi_y^j$]. We will denote $\int_x\equiv\int d^dx$ where d is the space dimension, and $\int_q\equiv\int d^dq/(2\pi)^d$ for integration in Fourier. Our aim is to compute the effective action $\Gamma(\phi)$, i.e., the generating function of proper vertices, since once it is known, all correlation functions are known being simply obtained as sums of all tree diagrams drawn using Γ. For all observables to be well defined, one usually requires both an ultraviolet (UV) cutoff (e.g., Λ_0 in momentum space) and an infrared (IR) cutoff (noted here $\Lambda_l=e^{-l}\Lambda_0$). For example, in a single scalar theory one chooses $G\equiv G_l$ with

$$G_l^q=q^{-2}c\left(\frac{q^2}{2\Lambda_l^2},\frac{q^2}{2\Lambda_0^2}\right) \tag{2}$$

in Fourier. Here $c(z,s)$ is a cutoff function which decreases to zero as $z\to 0$ or $s\to\infty$ and for convenience, see below, we choose $c(z,z)=0$. To study finite momentum observables in a massless theory, one is also interested in the zero IR cutoff limit, $\Lambda_l=0$ with $G\equiv G_{l=\infty}$ denoting $c(z)=c(\infty,z)$.

In this paper we will use that $\Gamma(\phi)$ satisfies the following exact RG functional equation when the quadratic part G is varied [for a fixed $\mathcal{V}(\phi)$]:

$$\partial\Gamma(\phi)=\frac{1}{2}\mathrm{Tr}\partial G^{-1}:\left[\frac{\delta^2\Gamma(\phi)}{\delta\phi\delta\phi}\right]^{-1}+\frac{1}{2}\phi:\partial G^{-1}:\phi. \tag{3}$$

Derivations and more details are given in Appendix A. This can be used to express how the effective action $\Gamma(\phi)\equiv\Gamma_l(\phi)$ of model (1) with $G\equiv G_l$ depends on the IR cutoff Λ_l. Indeed, the following property holds:

$$\Gamma_l(\phi)=-\frac{1}{2}\mathrm{Tr}\ln G_l+\frac{1}{2}\phi:G_l^{-1}:\phi+\mathcal{U}_l(\phi), \tag{4}$$

where $\mathcal{U}_l(\phi)\equiv\mathcal{U}_{G_l}(\phi)$ satisfies the exact flow equation,

$$\partial_l\mathcal{U}_l(\phi)=\frac{1}{2}\mathrm{Tr}\partial_lG_l:\left[G_l^{-1}-G_l^{-1}\left(1+G_l:\frac{\delta^2\mathcal{U}_l}{\delta\phi\delta\phi}\right)^{-1}\right] \tag{5}$$

with the initial condition $\mathcal{U}_{l=0}(\phi)=\mathcal{V}(\phi)$, simply reflecting that the effective action equals the action when all fluctuations are suppressed [at $l=0$ where the running propagator satisfies $G_{l=0}=0^+$ from the property $c(z,z)=0$]. The above equation (3) simply expresses how $\Gamma(\phi)$ in Eq. (4) depends on the final value $G\equiv G_l$. The zero IR cutoff limit $\Lambda_l=0$ can then be studied by integrating the above equation up to $l=\infty$.

For actual calculations, simpler and useful choices read, in momentum space,

$$G_l^q=\frac{1}{q^2}[c(q^2/2\Lambda_0^2)-c(q^2/2\Lambda_l^2)], \tag{6}$$

where the cutoff function $c(x)$ satisfies $c(0)=1$ and $c(\infty)=0$. With the choice $c(x)=1/(1+2x)$ one finds the massive, Pauli-Villars-like, propagator,

$$G_l^q=\frac{1}{q^2+m^2}-\frac{1}{q^2+M^2} \tag{7}$$

with $m=\Lambda_l^2$, $M=\Lambda_0^2$, where the IR cutoff mass $m\equiv m_l$ is lowered from $m=\infty$ ($l=0$) to $m=0$ ($l=\infty$) (and $\partial_l\to -m\partial_m$). Whenever one needs a stronger UV cutoff, one may use

$$G_m^q=\frac{1}{q^2+m^2}c\left(\frac{q^2}{2\Lambda_0^2}\right), \tag{8}$$

a different choice.

The full exact RG equation (5) can also be expanded in series of $\mathcal{U}_l$ as

$$\partial_l\mathcal{U}_l(\phi)=\frac{1}{2}\mathrm{Tr}\partial_lG_l:\frac{\delta^2\mathcal{U}_l(\phi)}{\delta\phi\delta\phi}-\frac{1}{2}\mathrm{Tr}\partial_lG_l:\frac{\delta^2\mathcal{U}_l(\phi)}{\delta\phi\delta\phi}:G_l:\frac{\delta^2\mathcal{U}_l(\phi)}{\delta\phi\delta\phi}+O(\mathcal{U}_l^3), \tag{9}$$

which admits the graphical representation given in Fig. 1.

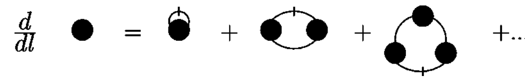

FIG. 1. Representation of the exact RG equations (3) and (9). The dot is the vertex $\mathcal{U}_l$, the solid line a propagator G_l, and the crossed solid line the on shell propagator ∂G_l. The sum is over all one loop graphs with a factor $(-1)^{p-1}/2$ for each p vertex graph represented.

To summarize, the philosophy of the method is, in a sense, the exact opposite of the Wilson one, since it amounts to start from the action with no fluctuations $\Lambda_l=\Lambda_0$, and then add modes and their fluctuations until one reaches the desired theory $\Lambda_l\ll\Lambda_0$. In that limit one expects that the effective action reaches a fixed point form, given by the asymptotic solution of Eq. (5) at large l.

B. Multilocal expansion

To handle the formidably complicated functional equation (5) we follow the method introduced in Ref. [1] and expand the interaction functional $\mathcal{U}_l$ in local, bilocal, trilocal, etc., components as

$$\mathcal{U}_l(\phi)=\int_x U_l(\phi_x)+\int_{xy}V_l(\phi_x,\phi_y,x-y)+\int_{xyz}W_l(\phi_x,\phi_y,\phi_z,x,y,z)+\cdots. \tag{10}$$

The local part depends only on the function $U_l(\phi)$, uniquely defined from the projection operator $\overline{P}_1$. This operator is fully defined in Ref. [1] (see also Appendix B). We recall here only its action on a bilocal operator $F(\phi_x,\phi_y,x-y)$, namely, $(\overline{P}_1 F)(\phi)=\int_y F(\phi,\phi,y)$. It can be used to split an action depending only on two points into

$$\int_{xy}F(\phi_x,\phi_y,x-y)=\int_x(\overline{P}_1 F)(\phi_x)+\int_{xy}[(1-P_1)F](\phi_x,\phi_y,x-y), \tag{11}$$

where, by definition, $(P_1 F)(\phi,\psi,z)=\delta(z)\int_y F(\phi,\psi,y)$, in such a way that the second part is properly bilocal [i.e., $(\overline{P}_1(1-P_1)F)(\phi)=0$]. A similar construction holds for higher multilocal operators.

The idea is then to project the functional equation (5) so that the bilocal, trilocal, etc., can be expressed exactly in terms of the local part U_l only. One notices that there is a simplest way to do it so that the bilocal part is $V\sim O(U^2)$, trilocal $W\sim O(U^3)$, etc. This determines one possible splitting of the higher multilocal components (e.g., bilocal versus trilocal) as is represented in Fig. 2, and further explained in Ref. [1]. This expansion is clearly suited to the situations where the flowing functional $\mathcal{U}_l$ becomes "small" and dominated by its local part (e.g., in the context of a dimensional expansion), but it has a more general validity, since in all cases it is an exact expansion in series of the local part of the full effective action functional.

We now pursue the analysis exactly to order $O(U_l^2)$, sufficient to a number of one loop applications. Details are given in Appendix B. The bilocal part is exactly given by

$$V_l(\phi_1,\phi_2,x)=\frac{1}{2}\left(F_l(\phi_1,\phi_2,x)-\delta(x)\int_y F_l(\phi_1,\phi_2,y)\right) \tag{12}$$

with

$$F_l(\phi_1,\phi_2,x)=-\int_0^l dl'(\partial^1\cdot\partial G^x_{l'}\cdot\partial^2)(\partial^1\cdot G^x_{l'}\cdot\partial^2)\times e^{-(1/2)\partial^1\cdot G^{x=0}_{l'l}\cdot\partial^1-(1/2)\partial^2\cdot G^{x=0}_{l'l}\cdot\partial^2-\partial^1\cdot G^x_{l'l}\cdot\partial^2}\times U_{l'}(\phi_1)U_{l'}(\phi_2) \tag{13}$$

to all orders (by definition), and the resulting exact RG equation for the local part of the effective action [i.e., the exact β function up to $O(U_l^3)$] is

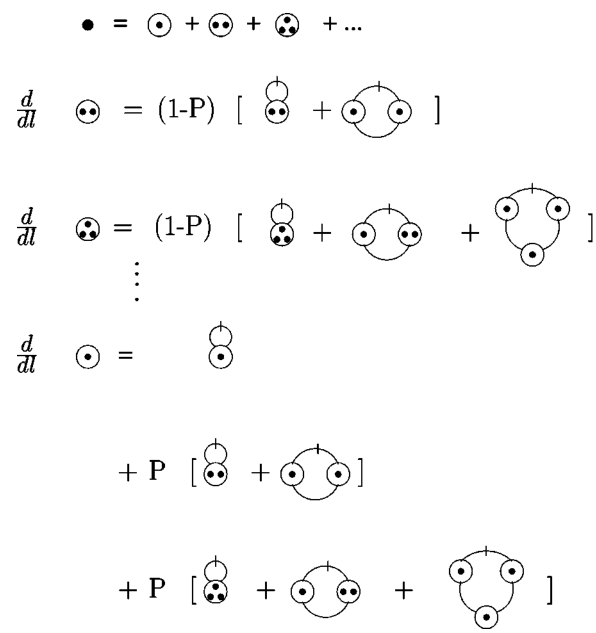

FIG. 2. Schematic representation of the splitting of the functional $\mathcal{U}$ vertex into local, bilocal, trilocal, etc., parts, respectively (top line). Representation of the exact RG equation for the bilocal, trilocal, etc., as well as local vertex (last several lines). Note that by definition the ERG equation for the bilocal part contains only exactly two feeding terms, trilocal three, etc. The solid lines represent a propagator G_l and the crossed solid lines the on shell propagator ∂G_l. Combinatorial factors are not represented. P here denotes the projection operator on the local part (denoted $\overline{P}_1$ and P_1 in the text).

$$\begin{aligned}\partial_l U_l(\phi)=&\frac{1}{2}\partial G^{x=0}_{l,ij}\partial_i\partial_j U_l(\phi)\\&-\frac{1}{2}\int_x \partial G^x_{l,ij}\partial_j\partial_k U_l(\phi)(G^x_l)_{km}\partial_m\partial_i U_l(\phi)\\&-\frac{1}{2}\int_x\partial^1\cdot(\partial G^x_l-\partial G^0_l)\cdot\partial^2\int_0^l dl'(\partial^1\cdot\partial G^x_{l'}\cdot\partial^2)\\&\times(\partial^1\cdot G^x_{l'}\cdot\partial^2)\\&\times e^{-(1/2)\partial^1\cdot G^{x=0}_{l'l}\cdot\partial^1-(1/2)\partial^2\cdot G^{x=0}_{l'l}\cdot\partial^2-\partial^1\cdot G^x_{l'l}\cdot\partial^2}\\&\times U_{l'}(\phi_1)U_{l'}(\phi_2)|_{\phi_1=\phi_2=\phi}.\end{aligned} \tag{14}$$

We use the following notations: $\partial_i\equiv\partial_{\phi^i}$, ∂_i^1 (∂_i^2) denotes derivation with respect to the first argument (second argument) of a function of two vectors ϕ_1, ϕ_2, $\partial^1\cdot G^{xy}\cdot\partial^2\equiv\Sigma_{ij}G^{xy}_{ij}\partial_i^1\partial_j^2$, etc. Also one notes in real space $G^{xy}\equiv G^{x-y}$ and $G^x_{l'l}=G^x_{l'}-G^x_l=-\int_{l'}^l\partial G^x_{l''}dl''$. Note that the first line contains two one loop diagrams (tadpole and bubble) with one "on shell" propagator, and the second line represents a sum over diagrams with at least two loops.

III. APPLICATION TO THE $O(N)$ MODEL

We first illustrate the method on the $O(N)$ model defined by Eq. (1) with

$$\mathcal{V}(\phi)=\frac{g_2}{2}\int_x \phi_x^2+\frac{g_4}{4!}\int_x (\phi_x^2)^2, \tag{15}$$

ϕ_x being a N-component vector, $\phi_x^2=\Sigma_i(\phi_x^i)^2$. The propagator is diagonal, and using an infrared cutoff Λ_l, it reads $G\equiv G_l$ with

$$G_{l,ij}^q=\delta_{ij}G_l^q \tag{16}$$

with G_l^q as in Eq. (2). We study this model near the dimension 4, in $d=4-\epsilon$, and compute the effective action to order $O(\epsilon^2)$. For some explicit calculations, we will further use the form (6) with the following convenient parametrization and notation for the cutoff function $c(z)$:

$$c(z)=\int_0^{+\infty} da\hat{c}(a)e^{-az}\equiv\int_a e^{-ax}. \tag{17}$$

The condition $c(0)=1$ imposes $\int_a=1$.

A. Derivation of the β functions and fixed points

The local part of the running effective action admits the polynomial expansion:

$$U_l(\phi)=g_{0,l}+\frac{g_{2,l}}{2!}\phi^2+\frac{g_{4,l}}{4!}(\phi^2)^2+\frac{g_{6,l}}{6!}(\phi^2)^3+\cdots. \tag{18}$$

From power counting, it is more convenient to introduce the dimensionless couplings $\tilde{g}_{2n,l}$ defined from

$$g_{2,l}=\Lambda_l^2\tilde{g}_{2,l},$$

$$g_{4,l}=\Lambda_l^{\epsilon}\tilde{g}_{4,l} \tag{19}$$

and more generally $\tilde{g}_{2n,l}=g_{2n,l}\Lambda_l^{(d-2)n-d}$ which flows to some fixed point values $\tilde{g}_{2n}^*$, as discussed below. Since $\tilde{g}_6^*\sim O(\epsilon^3)$ and $\tilde{g}_{2n}^*\sim O(\epsilon^n)$ for $n\geqslant 3$ (see Appendix C for the RG equation of $\tilde{g}_{6,l}$ and the free energy $\tilde{g}_{0,l}$), we drop from now on these higher monomials and study only the coupled RG equation for $\tilde{g}_4$ and $\tilde{g}_2$ easily obtained by inserting Eq. (18) into Eqs. (14) as detailed in Appendix C,

$$\partial_l\tilde{g}_{4,l}=\epsilon\tilde{g}_{4,l}-\frac{N+8}{3}\tilde{I}_l^{(1)}\tilde{g}_{4,l}^2=-\beta[\tilde{g}_{4,l}], \tag{20}$$

$$\partial_l\tilde{g}_{2l}=2\tilde{g}_{2l}+\frac{N+2}{6}\tilde{I}_l^{(0)}\tilde{g}_{4,l}-\frac{N+2}{3}\tilde{I}_l^{(1)}\tilde{g}_{2,l}\tilde{g}_{4,l} -\frac{N+2}{3}\int_0^l dl'\tilde{I}_{l,l'}^{(2)}\tilde{g}_{4,l'}^2, \tag{21}$$

with the integrals

$$\tilde{I}_l^{(0)}=\Lambda_l^{-2+\epsilon}\int_q\partial_lG_l^q, \quad \tilde{I}_l^{(1)}=\Lambda_l^{\epsilon}\int_q\partial_lG_l^qG_l^q. \tag{22}$$

$\tilde{I}_{l,l'}^{(2)}$ is given in Eq. (C7), where we show that the coefficient of the term proportional to $\tilde{g}_{4,l'}^2$ in Eq. (21) is well defined in the limit $l\to\infty$. One finds that $\tilde{I}_l^{(0)}=\tilde{I}^{(0)}$ is l independent and that $\lim_{l\to\infty}\tilde{I}_l^{(1)}=I^{(1)}$ is universal [independent of $c(s)$] in dimension $d=4$,

$$\tilde{I}^{(0)}=-\frac{1}{4\pi^2}\int_0^\infty dssc'(s)+O(\epsilon), \tag{23}$$

$$\tilde{I}^{(1)}=S_d\int_{s>0}(2s)^{-\epsilon/2}c'(s)[c(s)-1]=\frac{1}{16\pi^2}+O(\epsilon),$$

where S_d is the unit sphere area divided by $(2\pi)^d$ and we recall $c'(s)<0$. Finally, Eqs. (20) and (21) together with Eq. (22) lead to the fixed point values

$$\tilde{g}_4^*=\frac{48\pi^2}{N+8}\epsilon+O(\epsilon^2), \tag{24}$$

$$\tilde{g}_2^*=-\frac{N+2}{12}\tilde{I}^{(0)}\tilde{g}_4^*+O(\epsilon^2). \tag{25}$$

This fixed point describes the standard $O(N)$ critical system exactly at the critical temperature $T=T_c$. The initial conditions which end up for $l=\infty$ exactly at the fixed point describe the critical manifold.

Besides, we obtain the correction of the critical exponent characterizing the divergence of the magnetic susceptibility near the critical temperature from the positive eigenvalue λ_l (corresponding to the instable direction),

$$\partial_l(\tilde{g}_{2l}-\tilde{g}_2^*)=\lambda_l(\tilde{g}_{2l}-\tilde{g}_2^*), \tag{26}$$

$$\lambda_l=2\left(1-\frac{N+2}{2(N+8)}\epsilon\right), \tag{27}$$

which gives correctly [35] the exponent γ to order ϵ,

$$\gamma=\frac{2}{\lambda^*}=1+\frac{N+2}{2(N+8)}\epsilon+O(\epsilon^2). \tag{28}$$

B. Computation of the two and four point proper vertices

We now compute the effective action on the critical manifold, up to order $O(\epsilon)$ for the local part, and $O(\epsilon^2)$ for the bilocal part (i.e., the q dependent part), in the limit of large l. Equation (12) allows to construct the bilocal term in the effective action by inserting Eq. (18) in Eq. (13). As we restrict our analysis to order ϵ^2, we do not consider monomials higher than $(\phi^2)^2$ in Eq. (18), and therefore we expand the exponential in Eq. (13) to order 1. Using the combinatorics already explained for the local part in Appendix C, one gets

$$V_l(\phi_1,\phi_2,q)=\frac{1}{2}\int_x(e^{iqx}-1)F_l(\phi_1,\phi_2,x), \tag{29}$$

$$F_l(\phi_1,\phi_2,x)=\frac{N+2}{3}\phi_1\phi_2\int_0^l dl'\,\partial_{l'}G^x_{l'}G^x_{l'}G^x_{l'l}\tilde{g}^2_{4l'}\Lambda^{2\epsilon}_{l'}$$
$$-\left(\frac{N+4}{(3!)^2}\phi_1^2\phi_2^2+\frac{4}{(3!)^2}(\phi_1\cdot\phi_2)^2\right)$$
$$\times\int_0^l dl'\,\partial_{l'}G^x_{l'}G^x_{l'}\tilde{g}^2_{4l'}\Lambda^{2\epsilon}_{l'},$$

where we have not written terms of the form $f(\phi_i,x)$ (i.e., which depend only on one field argument) as they cancel out from the effective action. To this order in ϵ ($O(\epsilon^2)$) there are no other contributions. The explicit expressions of G_l^x and $\partial_l G_l^x$ using Eq. (17) are given in Appendix C [Eq. (C10)]. This bilocal term (29) allows to treat the renormalization of the wave function and compute the exponent η to order ϵ^2. A natural way to obtain it, within this method, is to compute directly the one particle irreducible (1PI) two-point function and then take the limit $l\to\infty$ (directly at T_c). Its local part comes from the quadratic contribution of Eq. (18) and the bilocal part is the sum of $G_l^{-1}(q)$ (4) and the quadratic contribution of Eq. (29),

$$\Gamma^{(2)}_{l,ij}(q)=\left.\frac{\delta^2\Gamma_l}{\delta\phi^i_q\delta\phi^j_{-q}}\right|_{\phi=0}=\delta_{ij}\Gamma^{(2)}_l(q),\qquad(30)$$

$$\Gamma^{(2)}_l(q)=G_l^{-1}(q)+\Lambda_l^2\tilde{g}_{2,l}-\frac{N+2}{18}\tilde{g}^2_{4,l}\int_x(e^{iqx}-1)(G_l^x)^3.$$

In Appendix C, we show that it has the form, up to terms of order $(\Lambda_l/\Lambda_0)^2$,

$$\Gamma^{(2)}_l(q)=G_l^{-1}(q)+\Lambda_l^2\tilde{g}_{2,l}-q^2\eta[\tilde{g}_{4,l}]\left[\ln\frac{\Lambda_l}{\Lambda_0}+\chi^{(2)}\left(\frac{q}{\Lambda_l}\right)\right],$$

$$\eta[\tilde{g}_{4,l}]=\frac{N+2}{18(4\pi)^4}\tilde{g}^2_{4,l}\qquad(31)$$

with the following asymptotic behaviors:

$$\chi^{(2)}(k)\sim ak^2,\quad k\ll1,$$
$$\chi^{(2)}(k)\sim\ln k,\quad k\gg1\qquad(32)$$

with a some nonuniversal [i.e., dependent on the cutoff function (17)] coefficient. The two-point scaling function $\chi^{(2)}(k)$ which is computed here (see Appendix C) for an arbitrary infrared cutoff function $c(x)$, is up to an additive constant, independent of the UV cutoff [61]. For the particular choice (7) one recovers the result of Ref. [36].

The large argument behavior of $\chi^{(2)}(k)$ allows to take the limit $l\to\infty$, using the fixed point value $\tilde{g}_4^*$ (24), we have (for $q\ll\Lambda_0$)

$$\lim_{l\to\infty}\Gamma^{(2)}_l(q)=\left(q^2-q^2\frac{N+2}{2(N+8)^2}\epsilon^2\ln\frac{q}{\Lambda_0}\right),\qquad(33)$$

which coincides with the expansion of $\lim_{l\to\infty}\Gamma^{(2)}_l(q)\sim q^2(q/\Lambda_0)^{-\eta}$ to order ϵ^2 with the universal value of the η exponent to this order,

$$\eta=\eta[\tilde{g}_4^*]=\frac{N+2}{2(N+8)^2}\epsilon^2,\qquad(34)$$

in agreement with standard results [35].

Let us focus on the construction of the quartic term in $\Gamma_l(\phi)$, obtained from the quartic contribution of Eqs. (18) and (29). After combinatorial manipulations, we obtain

$$\Gamma_l^{\rm quart}=\frac{\tilde{g}_{4l}\Lambda_l^\epsilon}{4!}\int'_{q_i}(\phi_{q_1}\cdot\phi_{q_2})(\phi_{q_3}\cdot\phi_{q_4})$$
$$-\tilde{g}_{4l}^2\frac{1}{(3!)^2}\int'_{q_i}\left[\left(\frac{N+4}{4}\right)(\phi_{q_1}\cdot\phi_{q_2})(\phi_{q_3}\cdot\phi_{q_4})\right.$$
$$\left.+(\phi_{q_1}\cdot\phi_{q_3})(\phi_{q_2}\cdot\phi_{q_4})\right]\chi_l^{(4)}(q_3+q_4)\qquad(35)$$

with $\chi_l^{(4)}(q)$ defined by

$$\chi_l^{(4)}(q)=\int_x(e^{iqx}-1)(G_l^x)^2+O(\Lambda_0^{-2})\qquad(36)$$

and where we used the notation $\int'_{q_i}\equiv\int_{q_1,q_2,q_3,q_4}(2\pi)^d\delta^{(d)}(q_1+q_2+q_3+q_4)$. The local term, i.e., the first line in Eq. (35), contains a contribution of order ϵ^2 which is divergent in the limit $l\to\infty$. Indeed, expanding it to second order gives $\tilde{g}_{4,l}\Lambda_l^\epsilon=\tilde{g}_{4,l}(1+\epsilon\ln\Lambda_l)+O(\epsilon^3)$ and at first sight this term would lead to a divergent contribution in the limit $l\to\infty$. However, the analysis of $\chi_l^{(4)}(q)=\chi^{(4)}(q/\Lambda_l)$ shows the following asymptotic behaviors:

$$\chi^{(4)}(k)\sim bk^2,\quad k\ll1,\qquad(37)$$

$$\chi^{(4)}(k)\sim-\frac{1}{16\pi^2}\ln(k^2),\quad k\gg1\qquad(38)$$

with b a nonuniversal constant. When considering the large l limit of the effective action, we are interested in the large argument behavior of $\chi^{(4)}(k)$ (38). Using the fixed point value $\tilde{g}_4^*$ (24), one gets that this cancels exactly the divergence when $l\to\infty$ due to the local term. Thus, we obtain

$$\lim_{l\to\infty}\Gamma_l^{\rm quart}=\frac{2\pi^2\epsilon}{(N+8)}\left\{\Lambda_0^\epsilon\int'_{q_i}(\phi_{q_1}\cdot\phi_{q_2})(\phi_{q_3}\cdot\phi_{q_4})\right.$$
$$+\frac{4\epsilon}{(N+8)}\int'_{q_i}\left[\left(\frac{N+4}{4}\right)(\phi_{q_1}\cdot\phi_{q_2})(\phi_{q_3}\cdot\phi_{q_4})\right.$$
$$\left.\left.+(\phi_{q_1}\cdot\phi_{q_3})(\phi_{q_2}\cdot\phi_{q_4})\right]\ln\left(\frac{|q_3+q_4|}{\Lambda_0}\right)\right\},\qquad(39)$$

which is independent of Λ_0 to order $O(\epsilon^2)$. Note that in the large N limit one recovers correctly the "screened" [37] four point renormalized vertex $\sim \epsilon q^{\epsilon}$ (where q is the transfer momentum).

The result of this analysis is that we have constructed the large scale theory by obtaining directly a fixed point for the effective action, keeping the UV cutoff Λ_0 finite, which is the relevant object for statistical physics, and for an arbitrary cutoff function.

C. Relation with field-theoretical methods

It is interesting to make the connection with standard field-theoretical methods for critical phenomena. There one is usually interested in the limit $\Lambda_0 \to \infty$. Note that in this limit Eq. (33) diverges. It is, however, possible to define a "renormalized" effective action $\Gamma_R(\phi_R)$ which is well defined in that limit.

One can first check directly on Eq. (33) the standard Callan-Symanzik (CS) "bare" RG equation [16] for the physical correlation function of the massless theory at the fixed point,

$$\left(\Lambda_0 \frac{\partial}{\partial \Lambda_0} - \eta\right)(\lim_{l \to \infty} \Gamma_l^{(2)})(q) = 0 + O(\epsilon^3). \tag{40}$$

One can also connect to the CS equation for the renormalized theory. One defines

$$\Gamma_R(\phi) = \Gamma_l(\sqrt{Z}\phi), \tag{41}$$

where $Z \equiv Z(\Lambda_l/\Lambda_0, \tilde{g}_{4,l})$ is the so-called "wave-function renormalization" factor such that

$$\Gamma_R^{(2)}(q) = m_R^2 + q^2 + O(q^4). \tag{42}$$

Using Eq. (30) and noting that $G_l^{-1}(q) = -2\Lambda_l^2/c'(0) + Aq^2 + O((\Lambda_l/\Lambda_0)^2)$ with $A = c''(0)/2c'(0)^2$, one finds the renormalized mass $m_R^2 = (1/A)\Lambda_l^2(2/|c'(0)| + \tilde{g}_{2,l})$ and $Z = (1/A)[1 + \eta(\tilde{g}_{4,l})\ln(\Lambda_l/\Lambda_0)]$. One can see that up to higher order terms, Λ_l plays the role of the renormalized mass. From Eq. (31) one finds, to order $(\epsilon \tilde{g}_{4,l}^2, \tilde{g}_{4,l}^3)$,

$$m_R \partial_{m_R}|_{\Lambda_0} \ln Z\left(\frac{\Lambda_l}{\Lambda_0}, \tilde{g}_{4,l}\right) = -\partial_l \ln Z = \eta[\tilde{g}_{4,l}], \tag{43}$$

these derivatives being taken at fixed $\tilde{g}_{4,l}$. This is the standard definition for the $\eta(g)$ function. One can go further, define a renormalized coupling g_R, e.g., through $\Gamma_R^{(4)}(q=0) = m_R^{\epsilon} g_R$, with $g_R = \tilde{g}_{4,l}$ up to higher order terms, and derive the CS equations for the renormalized vertices. Here, we just mention one such equation [16] for the "renormalized" two-point vertex function in the critical regime $\Lambda_l \ll \Lambda_0$ but *finite*

$$(\partial_l + \eta[\tilde{g}_{4,l}])\Gamma_l^{R(2)}(q) \simeq 0, \quad q/\Lambda_l \gg 1 \tag{44}$$

obtained using the large k behavior of $\chi^{(2)}(k)$ (32). We get again the universal value of the η exponent form $\eta = \eta[\tilde{g}_4^*]$ (34).

The connection between the EMRG method and the standard field theoretical methods in the *massless* scheme [i.e., imposing $\Gamma_R^{(2)}(q=0)=0$] is more subtle here (since one should use $l=\infty$ strictly).

IV. CARDY-OSTLUND MODEL: STATICS

In this section, we show how this EMRG method can be used to study perturbatively the Cardy-Ostlund model [25] near its glass transition.

A. Model, choice of propagator

This model is a random phase sine Gordon model which can represent an XY model in a random magnetic field where the vortices are excluded by hand. As mentioned in the Introduction, the statics of this model has been extensively studied using various methods [19,26–31]. The system at equilibrium is described by the partition function $Z = \int D\phi e^{-H^{\rm CO}[\phi]/T}$, T being the temperature with the Hamiltonian

$$H^{\rm CO}[\phi] = \frac{1}{2}\int d^2x(\nabla\phi_x)^2 - \int d^2x(h_x^1 \cos\phi_x + h_x^2 \sin\phi_x) \tag{45}$$

with $\phi_x \in]-\infty, +\infty[$ as there are no vortices, where $\mathbf{h}_x = (h_x^1, h_x^2)$ is a two dimensional random Gaussian vector of zero average with fluctuations decorrelated from site to site:

$$\langle h_x^i h_{x'}^j \rangle = 2g_0\Lambda_0^2 \delta_{ij}\delta^{(2)}(x-x'). \tag{46}$$

The quenched average over this random variable is performed by the means of replicas, which is used here as a simple trick to restore translational invariance and to organize perturbation theory. After averaging over the disorder, one obtains

$$\overline{\ln Z} = \lim_{n\to 0}\frac{\overline{Z^n}-1}{n}, \quad \overline{Z^n} = \int D\phi^a e^{-H^{\rm rep}[\phi^a]/T},$$

$$\frac{H^{\rm rep}[\phi^a]}{T} = \frac{1}{2T}\sum_{a,b}\int d^2x \nabla\phi_x^a \nabla\phi_x^b \delta_{a,b} - \frac{g_0\Lambda_0^2}{T^2}\sum_{ab}\int_x \cos(\phi_x^a - \phi_x^b), \tag{47}$$

where $a,b = 1,\dots,n$ are replica indices. We use the same propagator as for the $O(N)$ model, the Gaussian part of Eq. (47) being diagonal in replicas, one has

$$G_{l\,ab}^q = \delta_{ab}\frac{T}{q^2}[c(q^2/2\Lambda_0^2) - c(q^2/2\Lambda_l^2)] \tag{48}$$

with the same decomposition of the cutoff function $c(x)$ (17). Notice that the Hamiltonian $H^{\rm rep}$ possesses the statistical tilt symmetry (STS) [38]: the last term in Eq. (47) is invariant under the change of variable $\phi_x^a \to \phi_x^a + u_x$ which protects the diagonal (in replica space) quadratic term in the effective action to all orders in perturbation theory [29,34].

B. β functions and fixed point

For this model, the Fourier representation in the fields (B4) is more natural. Although only one harmonic is present in the starting Hamiltonian (47), higher harmonics are generated by perturbation theory and we write the local interacting part of the effective action (10) as

$$U_l(\phi) = -\Lambda_l^2 \sum_{K\neq 0} \frac{g_l^K}{T^2} e^{iK\cdot\phi}, \tag{49}$$

where $K=(K_1, \dots, K_n)$, $\phi=(\phi^1, \dots, \phi^n)$ are n-component vectors, and one defines $K\cdot K' = \Sigma_a K_a K'_a$. The sum is over all K such that K_a are integers, not all zero with $\Sigma_a K_a = 0$. $U_l(\phi)$ is real, imposing $g_l^K = g_l^{-K}$, and the symmetry under replica indices permutation, which is assumed here, imposes $g_l^K = g_l^{\sigma(K)}$, $\sigma(K)$ being any vector obtained from K by a permutation of the K_a. By inserting Eq. (49) in Eq. (14) [see also Eq. (B6) in Appendix B] one obtains the RG equation for the local part to second order in g_K,

$$\partial_l g_l^K = \left(2 - \frac{TK^2}{4\pi}\right) g_l^K + \frac{\tilde{J}_l^{(1)}}{2T^2} \sum_{P,Q,P+Q=K} g_l^P g_l^Q (P\cdot Q)^2$$
$$- \frac{1}{2T^2} \sum_{P,Q,P+Q=K} (P\cdot Q)^3 \int_0^l dl' \tilde{J}_{l,l'}^2 g_{l'}^P g_{l'}^Q \tag{50}$$

with the integrals

$$\tilde{J}_l^1 = \Lambda_l^2 \int_x \partial G_l^x G_l^x, \tag{51}$$

$$\tilde{J}_{l,l'}^2 = \Lambda_l^{-2} \int_x (\partial G_l^x - \partial G_l^{x=0}) \partial G_{l'}^x G_{l'}^x \Lambda_{l'}^4$$
$$\times e^{(P^2/2 + Q^2/2) G_{l'l}^{x=0} + P\cdot Q G_{l'l}^x}. \tag{52}$$

The glass transition temperature T_c below which the charges of minimal modulus such that $K_{1,-1} = (0, \dots, 1, \dots, -1, \dots, 0)$, $K_{1,-1}^2 = 2$ become relevant is

$$T_c = \frac{8\pi}{K_{1,-1}^2} = 4\pi \tag{53}$$

and a small parameter $\tau = (T_c - T)/T_c > 0$ can be defined, which allows to construct perturbatively the effective action of this model (47) in its glass phase. Indeed, just below T_c the higher harmonics are irrelevant [the eigenvalues $(2 - TK^2/4\pi)$ are negative and of order 1]. Such irrelevant higher harmonics include, for instance three replica term [62] $g_l^{1,-2,1} \Sigma_{a\neq b\neq c} e^{i(\phi_x^a - 2\phi_x^b + \phi_x^c)}$, corresponding to $K_{1,-2,1}^2 = 4$. We denote $g_l = g_l^{1,-1}$ the coupling constant associated with $K_{1,-1}$, and obtain its RG flow from Eq. (50) by taking into account the $2(n-2)$ possible fusions such that $P+Q = K_{1,-1}$, P,Q being themselves obtained by a permutation of the components of $K_{1,-1}$ $(g_l^P = g_l^Q = g_l)$ with $PQ = -1$ [25]. After some transformations detailed in Appendix D, one obtains

$$\partial_l g_l = \left(2 - \frac{T}{2\pi}\right) g_l - \mathcal{B}_l g_l^2, \tag{54}$$

$$\mathcal{B}_l = 2\partial\gamma_0(0) \int_{\tilde{x}} \gamma_l(\tilde{x}) + \frac{2}{T_c} \int_{\tilde{x}} [\partial\gamma_0(\tilde{x}) - \partial\gamma_0(0)]$$
$$\times (e^{T_c \gamma_l(\tilde{x})} - 1) + O(\tau), \tag{55}$$

where we used the dimensionless variable $\tilde{x} = x\Lambda_l$ and defined

$$\partial G_{l'}^x = T \partial\gamma_{\mu = l - l'}(\tilde{x}),$$
$$G_{l'l}^x = -T \gamma_{\mu = l - l'}(\tilde{x}), \tag{56}$$

where the two functions $\partial\gamma_\mu(x)$ and $\gamma_\mu(x)$ are given in Eq. (D8).

As shown in Appendix D, we can transform the integral over $\tilde{x}$ in [Eq. (55)] and express its cutoff dependence in a simple way. One finds $\mathcal{B}_\infty = (4\pi/T_c^2)\exp[-(\gamma_E - \int_a \ln 2a)]$ yielding for $T < T_c$, the stable fixed point of the RG flow is given by

$$g^* = 8\pi \exp\left[\left(\gamma_E - \int_a \ln 2a\right)\right] \tau + O(\tau^2) \tag{57}$$

with $\tau = (T_c - T)/T_c$ and $\gamma_E = 0.577\,216$ the Euler constant.

C. Bilocal term and two-point correlation function

Equation (B5) allows to construct the bilocal term in the effective action to lowest order [i.e., $O(\tau^2)$] using a Fourier representation (B4),

$$V_l(\phi,\psi,x) = \sum_{K,P} \hat{V}_l^{K,Px} e^{iK\cdot\phi + iP\cdot\psi}. \tag{58}$$

Just below T_c, only the charges of minimal modulus $K_{1,-1}^2 = 2$ are relevant, therefore to this order the sums in Eq. (B5) are restricted to such harmonics. By inserting Eq. (49) into Eq. (B5), one has

$$\hat{V}_l^{K,Pq} = \frac{1}{2} \int_x (e^{iqx} - 1) \hat{F}_l^{K,Px},$$

$$\hat{F}_l^{K,Px} = -\frac{(K\cdot P)^2}{T^4} \int_0^l dl' \, \partial G_{l'}^x G_{l'}^x e^{(K^2+P^2)/2 G_{l'l}^{x=0}}$$
$$\times e^{K\cdot P G_{l'l}^x} \Lambda_{l'}^4 g_{l'}^K g_{l'}^P, \tag{59}$$

where K,P are of the form $K_{1,-1}$, and thus $g_l^K=g_l^P=g_l$. Performing the integral over l' as explained in Appendix D, we have

$$\hat{F}_l^{K,P\tilde{x}}=-\frac{\Lambda_l^4}{T^2}g_l^2\left(\frac{1}{T^2}(e^{-T_cK\cdot P\gamma_l(\tilde{x})}-1)+K\cdot P\frac{\gamma_l(\tilde{x})}{T}\right) \tag{60}$$

with $\tilde{x}=\Lambda_l x$. For the charges K,P we are considering here, there are *a priori* five different cases of $K\cdot P=-2,-1,0,1,2$ to consider. However, we see immediately in the previous expression (60) that the charges such that $K\cdot P=0$ do not contribute to the bilocal part of the effective action (they correspond to four replica terms $g_l^2\Sigma_{a\neq b\neq c\neq d}e^{i(\phi_a-\phi_b)+i(\psi_c-\psi_d)}$). We show in Appendix D that $\hat{V}_l^{K,P,q}$ takes the form, up to terms of order $(\Lambda_l/\Lambda_0)^2$,

$$\hat{V}_l^{K,P,q}=-A_lq^2\left[\delta_{K,-P}\ln\frac{\Lambda_l}{\Lambda_0}+\chi^{K,P}\left(\frac{q}{\Lambda_l}\right)\right], \tag{61}$$

$$A_l=\frac{\pi g_l^2}{4T_c^4}\exp\left(-2\gamma_E+2\int_a\ln 2a\right), \tag{62}$$

where $\chi^{K,P}(k)$ behaves asymptotically at small argument as

$$\chi^{K,P}(k)\sim\begin{cases}a_{K\cdot P}, & K\cdot P\neq-2\\ a_{-2}k^2, & K\cdot P=-2\end{cases}\quad k\ll 1 \tag{63}$$

and at large argument (relevant for the limit $l\to\infty$) as

$$\chi^{K,P}(k)\sim\begin{cases}b_{K\cdot P}\dfrac{1}{k^2}, & K\cdot P=1,2\\ b_{-1}\dfrac{\ln k}{k^2}, & K\cdot P=-1\\ \ln k, & K\cdot P=-2\end{cases}\quad k\gg 1. \tag{64}$$

The large argument behavior of $\chi^{K,P}(x)$ allows to take the limit $\Lambda_l\to 0$ of Eq. (61) as the logarithmic divergence (which only exists for $K=-P$) is canceled. We notice also that only such terms with $K=-P$ survive in this limit: in particular, three replica terms such as $g_l^2\Sigma_{a\neq b\neq c}e^{i(\phi_a-\phi_b)+i(\psi_b-\psi_c)}$ do not exist in the effective action to order τ^2 at the fixed point for $\Lambda_l=0$. Besides, by inserting the fixed point value g^* (57) in A_l (62), we see that the cutoff dependence [encoded in the factor $\exp(\int_a\ln 2a)$] disappears in $\lim_{l\to\infty}A_l$ leading to

$$\lim_{l\to\infty}\hat{V}_l^{K,Pq}=-\delta_{K,-P}\frac{\tau^2}{16\pi}q^2\ln\frac{q}{\Lambda_0}. \tag{65}$$

Equation (4), together with Eq. (65), allows to construct the bilocal term as

$$\lim_{l\to\infty}\Gamma_l^{biloc}(\phi)=\frac{1}{2T}\sum_a\int_q\frac{q^2}{c\left(\dfrac{q^2}{2\Lambda_0^2}\right)}\phi_q^a\phi_{-q}^a+\sum_{a,b}\int_{x,x'}\int_q\lim_{l\to\infty}\hat{V}_l^{K,-Kq}\times e^{iq(x-x')}e^{i(\phi_x^a-\phi_x^b)}e^{-i(\phi_{x'}^a-\phi_{x'}^b)} \tag{66}$$

from which we extract the two-point 1PI function $\lim_{l\to\infty}\Gamma_{l,ab}^{(2)}(q)$,

$$\Gamma_{l,ab}^{(2)}(q)=\left.\frac{\delta^2\Gamma_l(\phi)}{\delta\phi_q^a\delta\phi_{-q}^b}\right|_{\phi=0}, \tag{67}$$

$$\lim_{l\to\infty}\Gamma_{l,ab}^{(2)}(q)=\frac{q^2}{Tc(q^2/2\Lambda_0^2)}\delta_{ab}+\frac{\tau^2}{4\pi}q^2\ln\frac{q}{\Lambda_0}, \tag{68}$$

from which we extract the correlation function at the fixed point (up to terms of order Λ_0^{-2})

$$\overline{\langle[\phi_x-\phi_0]^2\rangle}=\lim_{l\to\infty}2\int_q(1-e^{iqx})[\Gamma_l^{(2)}]_{aa}^{-1} \tag{69}$$

$$\begin{aligned}&=2T_c\int_q\frac{1-e^{iqx}}{q^2}c\left(\frac{q^2}{2\Lambda_0^2}\right)\times\left[1-\tau-\tau^2\ln\left(\frac{q}{\Lambda_0}\right)c\left(\frac{q^2}{2\Lambda_0^2}\right)\right]\\&\sim 2\tau^2\ln^2(|x|\Lambda_0)+4[1-\tau+O(\tau^2)]\times\ln(|x|\Lambda_0),\end{aligned} \tag{70}$$

which shows that the amplitude of these anomalous fluctuations in $\ln^2(|x|\Lambda_0)$ is universal (Ref. [34]).

We finally mention that, due to the STS, the connected correlation function $\overline{\langle[\phi_x-\phi_0]^2\rangle}-\overline{\langle[\phi_x-\phi_0]\rangle\langle[\phi_x-\phi_0]\rangle}$ is the same as in the pure system.

V. CARDY-OSTLUND: EQUILIBRIUM DYNAMICS

We now turn to dynamics, which, within the EMRG framework can be conveniently studied by introducing an infrared cutoff on space only, keeping the full time dependence.

A. Model and propagator

Within this EMRG framework we want to study the dynamics of the model (45) [32,33], described by a Langevin type equation,

$$\eta\frac{\partial}{\partial t}u_{xt}=-\frac{\delta H^{\rm CO}}{\delta u_{xt}}+\zeta(x,t), \tag{71}$$

where $\langle\zeta(x,t)\rangle=0$ and $\langle\zeta(x,t)\zeta(x',t')\rangle=2\eta T\delta(x-x')\delta(t-t')$ is the thermal noise and η the friction coefficient. A convenient way to study the dynamics is to use the Martin-Siggia-Rose (MSR) [39] generating functional, on which perturbation theory can be done. Moreover, using the Ito prescription, it can be readily averaged over the disorder. The disorder averaged generating functional reads

$$Z[j,\hat{j}]=\int \mathcal{D}u\mathcal{D}i\hat{u}e^{-S[u,i\hat{u}]+j:u+\hat{j}:i\hat{u}}, \tag{72}$$

$$S[u,i\hat{u}]=S_0[u,i\hat{u}]+S_{\rm int}[u,i\hat{u}],$$

$$S_0[u,i\hat{u}]=\int_{qt}i\hat{u}_{-qt}(\eta\partial_t+cq^2)u_{qt}-\eta T\int_{xt}i\hat{u}_{xt}i\hat{u}_{xt},$$

$$S_{\rm int}[u,i\hat{u}]=-g_0\Lambda_0^2\int_{xtt'}i\hat{u}_{xt}i\hat{u}_{xt'}\cos(u_{xt}-u_{xt'}),$$

where $\int_t=\int_{t_i}^{\infty}dt$, where in this section the initial time t_i is sent to $t_i=-\infty$ before taking Λ_l/Λ_0 large, in order to describe equilibrium dynamics.

In our formulation (1), the field ϕ is now a two-component vector

$$\phi_{xt}=\begin{pmatrix} u_{xt}\\ i\hat{u}_{xt}\end{pmatrix} \tag{73}$$

and from S_0 in Eq. (72), we compute the inverse bare propagator G_l^{-1},

$$G_l^{-1}(q)=\begin{pmatrix} 0 & \delta(t-t')(-\eta\partial_t+cq^2)\\ \delta(t-t')(\eta\partial_t+cq^2) & -2\eta T\delta(t-t')\end{pmatrix}$$
$$\times\frac{1}{c(q^2/2\Lambda_0^2)-c(q^2/2\Lambda_l^2)}. \tag{74}$$

By inverting this matrix we obtain the bare response and correlation functions

$$C_{l\,tt'}^{q}=C_{l\,t't}^{q}=\overline{\langle u_{qt}u_{-qt'}\rangle}$$
$$=\frac{T}{q^2}e^{-q^2|t-t'|}[c(q^2/2\Lambda_0^2)-c(q^2/2\Lambda_l^2)], \tag{75}$$

$$R_{l\,tt'}^{q}=\overline{\frac{\delta\langle u_{qt}\rangle}{\delta h_{-qt'}}}=\overline{\langle u_{qt}i\hat{u}_{-qt'}\rangle}$$
$$=\theta(t-t')e^{-q^2(t-t')}[c(q^2/2\Lambda_0^2)-c(q^2/2\Lambda_l^2)], \tag{76}$$

where we have set the bare $\eta=1$. As we consider here the equilibrium dynamics of the system, the time translation invariance (TTI) and the fluctuation dissipation theorem (FDT) hold. These properties hold to all orders in perturbation theory and, as we will see, have strong consequences on the structure of the effective action $\Gamma_l(u,i\hat{u})$. This means for the dressed (i.e., exact) response and correlation functions:

$$\mathcal{C}_{l\,tt'}^{q}=\mathcal{C}_{l\,t-t'}^{q}, \tag{77}$$

$$\mathcal{R}_{l\,tt'}^{q}=\mathcal{R}_{l\,t-t'}^{q}, \tag{78}$$

$$\mathcal{R}_{l\,t-t'}^{q}=-\theta(t-t')\frac{1}{T}\partial_t\mathcal{C}_{l\,t-t'}^{q}. \tag{79}$$

B. Response function and dynamical exponent

We will study the dynamics near the transition temperature T_c (53), below which the lowest harmonic of the disordered potential becomes relevant. Near T_c, we showed previously that the higher harmonics, although generated by perturbation theory, are irrelevant.

As we considered here static disorder, the average over the disorder generates an effective interaction $S_{\rm int}[u,i\hat{u}]$ in Eq. (72), which is nonlocal in time, so we expect the friction coefficient to be renormalized by the disorder. We therefore construct the effective action to order 1 in $\tau=(T-T_c)/T_c$, and extract the dynamical exponent z from the response function. In the starting dynamical action (72), the interacting part is purely local in space, so to order 1 the interacting part of the associated effective action $\Gamma_l(u,i\hat{u})$ will remain so. We therefore search a perturbative solution of the equation for $\Gamma_l(u,i\hat{u})$ of the form (10),

$$\mathcal{U}_l(u,i\hat{u})=\int_x U_l(u_x,i\hat{u}_x) \tag{80}$$

$$=\int_{xt}i\hat{u}_{xt}F_{lt}(u_x)-\frac{1}{2}\int_{xtt'}i\hat{u}_{xt}\Delta_{ltt'}(u_x)i\hat{u}_{xt'}, \tag{81}$$

where $F_{lt}(u_x)$ and $\Delta_{ltt'}(u_x)$ are functionals only with respect to the time dependence, i.e., functions of the "vector" $u_x\equiv\{u_{xt}\}$ at a given point x in space. In addition, these will acquire an *explicit* time dependence, indicated by their t and t' indices. One has the initial conditions

$$\Delta_{l=0tt'}(u)=2g_0\Lambda_0^2\cos(u_t-u_{t'}), \tag{82}$$

$$F_{l=0t}(u)=0.$$

The $F_{lt}(u)$ term is indeed generated by perturbation theory and is related—in the case of equilibrium dynamics—to $\Delta_{ltt'}(u)$ by a generalized FDT relation, namely, a Ward identity, which can be written to lowest order,

$$\frac{\delta F_{lt}(u)}{\delta u_{xt'}}=-\frac{1}{T}\partial_{t'}\Delta_{ltt'}(u),\quad t>t', \tag{83}$$

where $\partial_{t'}$ acts only on the explicit time dependence (i.e., not on $u_{t'}$). Notice finally that terms containing higher powers of the field $i\hat{u}$, i.e., $(i\hat{u})^{p+2}$ are of order τ^{p+1}. They correspond to higher cumulant of the disorder (i.e., higher number of replica terms in the statics). The exact RG equation to order 1 (14) then reads (see Appendix E)

$$\partial_l \Delta_{ltt'}(u) = \int_{t_1 t_1'} k^{(1)}_{l t_1 t_1'} \Delta_{ltt'}(u), \tag{84}$$

$$\partial_l F_{lt}(u) = \int_{t_1 t_1'} k^{(1)}_{l t_1 t_1'} F_{lt}(u) - \int_{t_1 > t_1'} k^{(2)}_{l t_1 t_1'} \Delta_{ltt_1'}(u)$$

with

$$k^{(1)}_{l t_1 t_1'} = \frac{1}{2} \frac{\delta}{\delta u_{xt_1}} \partial_l C^{x=0}_{l t_1 t_1'} \frac{\delta}{\delta u_{xt_1'}},$$

$$k^{(2)}_{l t_1 t_1'} = \frac{\delta}{\delta u_{xt_1}} \partial_l R^{x=0}_{l t_1 t_1'}. \tag{85}$$

The solution of this coupled set of equations (84) together with Eq. (82) is given by

$$\Delta_{ltt'}(u) = 2\Lambda_l^2 g_l e^{C^{x=0}_{l t-t'}} \cos(u_{xt} - u_{xt'}), \tag{86}$$

$$\frac{\delta F_{lt}(u)}{\delta u_{xt'}} = -2\Lambda_l^2 g_l e^{C^{x=0}_{l t-t'}} R^{x=0}_{l t-t'} \cos(u_{xt} - u_{xt'}), \quad t > t',$$

where we can check explicitly the previously mentioned generalized FDT relation (83). Finally, as we consider here static disorder, the flow of g_l is given by the previous study, the fixed point value g^* being given by Eq. (57).

From $\Gamma_l[u, i\hat{u}]$, we obtain the response function in the following way:

$$\mathcal{R}^q_{l t-t'} = \langle u_{qt} i\hat{u}_{-qt'} \rangle = \left(\frac{\delta^2 \Gamma_l}{\delta i\hat{u}_{qt_1} \delta u_{-qt_1'}} \bigg|_{u = i\hat{u} = 0} \right)^{-1}_{q,t,t'}. \tag{87}$$

We define

$$D_{ltt'} = \Delta_{ltt'}(u=0), \tag{88}$$

$$\Sigma_{ltt'} = \frac{\delta F_{lt}(u)}{\delta u_{xt'}} \bigg|_{u=0}.$$

Notice that in the case of equilibrium dynamics $D_{ltt'} = D_{lt-t'}$ and $\Sigma_{ltt'} = \Sigma_{lt-t'}$. One gets

$$\frac{\delta^2 \Gamma_l}{\delta i\hat{u}_{qt} \delta u_{-qt'}} \bigg|_{u = i\hat{u} = 0} = \delta(t-t')(q^2 + \partial_t) + \Sigma_{lt-t'}. \tag{89}$$

When considering equilibrium dynamics, the use of Fourier transform allows to compute this matrix element (87) in a simple way,

$$\mathcal{R}^q_{l\omega} = \frac{1}{q^2 - i\omega + \Sigma_{l\omega}}, \tag{90}$$

where $\mathcal{R}^q_{lt} = \int d\omega/2\pi e^{-i\omega t} \mathcal{R}^q_{l\omega}$ and $\Sigma_{l\omega}$ is the Fourier transform, of $\Sigma_{lt-t'}$. In Appendix E we show that it has the following form (up to terms of order Λ_l^2/Λ_0^2):

$$\Sigma_{l\omega} = i\omega B_l \left[\ln \frac{\Lambda_l^2}{\Lambda_0^2} + \chi^{(\mathrm{dyn})}\left(\frac{\omega}{\Lambda_l^2} \right) \right], \tag{91}$$

$$B_l = \frac{g_l \exp\left(\int_a \ln 2a \right)}{2T_c}, \tag{92}$$

with the following asymptotic behaviors:

$$\chi^{(\mathrm{dyn})}(\nu) \sim a_{\mathrm{dyn}} \ln \nu, \quad \nu \ll 1, \tag{93}$$

$$\chi^{(\mathrm{dyn})}(\nu) \sim \ln \nu, \quad \nu \gg 1, \tag{94}$$

where a_{dyn} is a nonuniversal constant. The large argument behavior of $\chi^{(\mathrm{dyn})}$ (94) allows to take the large l limit in Eq. (91) as the logarithmic divergence is canceled, which gives

$$\lim_{l \to \infty} \mathcal{R}^q_{l\omega} = \frac{1}{q^2 - i\omega + i\omega B^* \ln \dfrac{\omega}{\Lambda_0^2}}, \tag{95}$$

$$B^* = \lim_{l \to \infty} B_l = e^{\gamma_E} \tau, \tag{96}$$

where we have used Eq. (57) to compute B^* which is universal: the cutoff dependence encoded in $\exp(2\int_a \ln 2a)$ has disappeared. On the other hand, we expect that the scaling function in Fourier should read

$$\lim_{l \to \infty} \mathcal{R}^q_{l\omega} = \frac{1}{q^2 - i\omega \left(\dfrac{\omega}{\Lambda_0^2} \right)^{2/z - 1}} \tag{97}$$

from scaling. If the initial model possess STS then the coefficient of q^2 is fixed to unity. The q independence of the self-energy is expected to hold only to the order in τ that we are working at, and it should be corrected by higher loops. Expansion of the denominator of Eq. (95) coincides with the expansion to order τ of the denominator of Eq. (97) and yields the universal value of the dynamical exponent z,

$$z - 2 = 2B^* = 2e^{\gamma_E}\tau + O(\tau^2) \tag{98}$$

in agreement with previous studies.

It is interesting in view of later applications to nonequilibrium dynamics, and a useful check, to compute this response function in the time domain. Indeed, writing simply the identity $\Gamma_l^{(2)} \Gamma_l^{(2)-1} = \mathbb{I}$, where $\Gamma_l^{(2)}$ is the matrix of the second functional derivatives of the effective action with respect ot the fields u_{xt} and $i\hat{u}_{xt}$, we obtain a system of closed equations for the exact response and correlation functions $\mathcal{R}^{xx'}_{ltt'}$ and $\mathcal{C}^{xx'}_{ltt'}$ to order 1 [more generally, $F_{lt}(u)$ and $\Delta_{ltt'}$ can be bilocal in space],

$$\partial_t \mathcal{R}_{ltt'}^{xx'} - \nabla^2 \mathcal{R}_{ltt'}^{xx'} + \int_{t_i}^{t} dt_1 \Sigma_{ltt_1} \mathcal{R}_{lt_1t'}^{xx'} = \delta(t-t')\,\delta(x-x'), \tag{99}$$

$$\partial_t \mathcal{C}_{ltt'}^{xx'} - \nabla^2 \mathcal{C}_{ltt'}^{xx'} + \int_{t_i}^{t} dt_1 \Sigma_{ltt_1} \mathcal{C}_{lt_1t'}^{xx'} = 2\eta T \mathcal{R}_{lt't}^{xx'} + \int_{t_i}^{t'} dt_1 D_{ltt_1} \mathcal{R}_{t't_1}^{xx'}. \tag{100}$$

We remind that we have chosen the Ito prescription, which fixes the following initial condition for the response function:

$$\lim_{\epsilon\to 0} \mathcal{R}_{lt,t-\epsilon} = 1,$$

$$\mathcal{R}_{lt,t} = 0. \tag{101}$$

Before using these equations to study nonequilibrium dynamics, we show how the equation for the response [Eq. (99)] function allows to recover the dynamical exponent z. Using Eq. (88) together with Eq. (86) and TTI (which holds for equilibrium dynamics), the equation for the response function reads

$$(\partial_t + q^2)\mathcal{R}_{lt-t'}^{q} = 2 g_l \Lambda_l^2 \int_{-\infty}^{t} dt_1 R_{lt-t_1}^{x=0} e^{C_{lt-t_1}^{x=0}} \times (\mathcal{R}_{lt_1-t'}^{q} - \mathcal{R}_{lt-t'}^{q}). \tag{102}$$

The limit $l\to\infty$ is taken as explained in Appendix E (E20), and a way to solve this equation is simply to say that in the right hand side (rhs), we may replace $\mathcal{R}_{lt_1t'}^{q}$, by its bare value, which is simply $\theta(t_1-t')e^{-q^2(t_1-t')}$ as this term is already of order τ.

One expects that the response function can be written as

$$\mathcal{R}_{t-t'}^{q} = \lim_{l\to\infty} \mathcal{R}_{l,t-t'}^{q} = \tilde{q}^{z-2} F_R^{\mathrm{eq}}(\tilde{q}^{z}(\tilde{t}-\tilde{t}')), \tag{103}$$

where $\tilde{q} = q/\Lambda_0$, $\tilde{t} = t\Lambda_0^2$, $\tilde{t}' = t'\Lambda_0^2$, with F_R^{eq} a universal scaling function (up to an overall nonuniversal scale) such that $F_R^{\mathrm{eq}}(v) \sim v^{(2-z)/z}$ for $v\to 0$. As a function it admits an expansion in powers of τ, obtained as

$$F_R^{\mathrm{eq}}(v) = F_R^{0}(v) + \tau F_R^{1\mathrm{eq}}(v) + O(\tau^2), \tag{104}$$

$$F_R^{0}(v) = e^{-v},$$

$$F_R^{1\mathrm{eq}}(v) = e^{\gamma_E}[(v-1)Ei(v)e^{-v} + e^{-v} - 1],$$

$$v = \tilde{q}^{z}(\tilde{t}-\tilde{t}'),$$

as shown in Appendix E. This is established by identifying the direct expansion of Eq. (103) in terms of the argument $v' = \tilde{q}^{2}(\tilde{t}-\tilde{t}')$,

$$\mathcal{R}_{\tilde{t}}^{\tilde{q}} = F_R^{0}(v') + (z-2)\ln\tilde{q}[F_R^{0}(v') + v' F_R^{0'}(v')] + \tau F_R^{1\mathrm{eq}}(v') + O(\tau^2) \tag{105}$$

with the result of solving Eq. (102). Note that the term proportional to $\ln\tilde{q}$ has precisely the expected v dependence, a check of the calculation. Since there is an overall nonuniversal scale $\tilde{q}\to\lambda\tilde{q}$, $F_R^{1\mathrm{eq}}(v)$ is defined up to a change in the constant ρ defined in Appendix E (E34).

One can check explicitly that the scaling function in the time domain obtained by this second method coincides with the inverse Fourier transform of Eq. (97) to the lowest order in τ. The asymptotic behavior of the scaling function in the time domain is

$$F_R^{1\mathrm{eq}}(v) \approx e^{\gamma_E} \ln[1/(e^{\gamma_E} v)], \quad v\to 0, \tag{106}$$

$$F_R^{1\mathrm{eq}}(v) \approx e^{\gamma_E} v^{-2}, \quad v\to\infty, \tag{107}$$

the slow time decay $1/t^{1+2/z}$, for $z>2$, arises from the disorder. Notice that a similar power law tail for large $\tilde{q}^{z}\tilde{t}$ has already been obtained for the diluted Ising model [40].

Using the FDT we also obtain the equilibrium correlation function in the scaling regime as

$$\mathcal{C}_{tt'}^{q} = T\tilde{q}^{-2} F_C^{\mathrm{eq}}(\tilde{q}^{z}(\tilde{t}-\tilde{t}')), \tag{108}$$

$$F_C^{\mathrm{eq}}(v) = \int_{v}^{+\infty} dw F_R^{\mathrm{eq}}(w). \tag{109}$$

We conclude this section on equilibrium dynamics by noticing a few interesting properties. The first one is an exact consequence of the scaling form (103) combined with the STS. Indeed, the STS imposes

$$\lim_{t\to\infty} \int_{\tilde{t}_i}^{\tilde{t}} dt' \mathcal{R}_{\tilde{t}\tilde{t}'}^{\tilde{q}} = \frac{1}{\tilde{q}^2}. \tag{110}$$

Using the scaling property we showed previously, this symmetry (110) implies

$$\int_0^{\infty} dt \tilde{q}^{z-2} F_R^{\mathrm{eq}}(\tilde{q}^{z} t) = \frac{1}{\tilde{q}^2} \Rightarrow \int_0^{\infty} du F_R^{\mathrm{eq}}(u) = 1, \tag{111}$$

from which it follows that

$$\mathcal{R}_{tt'}^{x=0} = \int_q \tilde{q}^{z-2} F_R^{\mathrm{eq}}(\tilde{q}^{z}(\tilde{t}-\tilde{t}')) = \frac{1}{2\pi z(t-t')}, \tag{112}$$

$$\mathcal{C}_{\tilde{t}\tilde{t}'}^{x=0} = \frac{T}{2\pi z}\ln(t-t'),$$

where we have used FDT in the last line. Note that the unrescaled time t appears in these formulas. Although the scaling form (103) is only valid for small $\tilde{q}$, we believe that the behaviors (113) may actually be the exact leading ones in the large $t-t'$ limit, their coefficients being fixed (nonperturbatively) by the STS. This would be interesting to check numerically.

The second property is a comparison with the so-called Porod's law [41]. If the form (97) were to hold to all orders, the scaling functions would decay at large arguments as $F_R^{\rm eq}(v)\sim 1/v^{1+2/z}$ and $F_C^{\rm eq}(v)\sim 1/v^{2/z}$. That yields

$$\mathcal{C}^q_{tt'}\sim\frac{1}{(t-t')^{2/z}q^4}\tag{113}$$

as in the Porod's law with $d=2$ and $n=2$ [41]. Here this property holds to the order of our calculation $O(\tau)$.

VI. NONEQUILIBRIUM DYNAMICS OF THE CO MODEL

Applying standard scaling arguments, we expect $\mathcal{R}^q_{tt'}$ and $\mathcal{C}^q_{tt'}$ to be functions of the scaling variables $\tilde{q}^z\tilde{t}$ and $\tilde{q}^z\tilde{t}'$ where $\tilde{q}=q/\Lambda_0$ and $\tilde{t}=\Lambda_0^2 t$ and z is the dynamical exponent. As is the case for pure systems at a critical point, one can write from RG arguments [42] with little restriction,

$$\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\tilde{q}^{-2+z+\eta}\left(\frac{\tilde{t}}{\tilde{t}'}\right)^{\theta}F_R(\tilde{q}^z(\tilde{t}-\tilde{t}'),\tilde{t}/\tilde{t}'),\tag{114}$$

$$\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=T\tilde{q}^{-2+\eta}\left(\frac{\tilde{t}}{\tilde{t}'}\right)^{\theta}F_C(\tilde{q}^z(\tilde{t}-\tilde{t}'),\tilde{t}/\tilde{t}'),\tag{115}$$

where the exponent θ is defined by imposing the following behavior of the response scaling function $F_R(v,u)$ when $u\to\infty$:

$$F_R(v,u)=F_{R,\infty}(v)+O(u^{-1}).\tag{116}$$

This has been checked for pure systems [42–45] and, partially for one case of a disordered system (only for the response function in Ref. [40] and for the Fourier mode $q=0$ for both functions in Ref. [46]). It was found in all the pure cases that one also has

$$F_C(v,u)=\frac{F_{C,\infty}(v)}{u}+O(u^{-2}).\tag{117}$$

These forms, Eqs. (116) and (117), yield a nontrivial fluctuation dissipation ratio (FDR) characterizing the violation of the FDT [47,48]. It has been computed exactly for the spherical model in $d>2$ [43], using dynamical RG methods for the pure $O(N)$ model at criticality up to two loops in an $\epsilon=4-d$ expansion [45], and up to one loop for the critical diluted Ising model in a $\sqrt{\epsilon}$ expansion [46].

Another standard definition for the autocorrelation exponent λ_C [49–51] and for the autoresponse exponent λ_R [52] is

$$\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\tilde{t}^{(d-\lambda_C)/z}\phi_C(\tilde{q}^z\tilde{t}),\tag{118}$$

$$\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\tilde{t}^{(d-\lambda_R/z)}\phi_R(\tilde{q}^z\tilde{t})\tag{119}$$

in the limit $\tilde{t}\to\infty$, $\tilde{q}\to 0$ with $\tilde{t}'$ fixed and $\tilde{q}^z\tilde{t}$ fixed, with $\phi_{R,C}(0)=$const. Assuming the behaviors (116) and (117) one finds the connection

$$(d-\lambda_C)z^{-1}=\theta-1+(2-\eta)z^{-1},\tag{120}$$

$$\lambda_R=\lambda_C,$$

$$\phi_C(v)=Tv^{(\eta-2)/z}F_{C,\infty}(v)(t')^{1-\theta},$$

$$\phi_R(v)=v^{(\eta-2+z)/z}F_{R,\infty}(v)(t')^{\theta},$$

which seems to hold for pure models, together with the inequality $d/2\leqslant\lambda_C=\lambda_R$ [49,53].

For the nonequilibrium dynamics of the CO model, we obtain similar scalings (114), (115), ($\eta=0$ in this case because of STS) but with a *different asymptotic behavior* at large u of the scaling function $F_C(v,u)$. As we will see, this has strong consequences on the FDR. Note that although $\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ is the full correlation function, to this order in the τ expansion it coincides with the connected one (which is the correct one to consider, e.g., to obey FDT in the equilibrium regime), the difference between the two being of order $g^2=O(\tau^2)$.

A. General framework

We want to study the dynamics of the system described by Eq. (71) which, at the initial time $t_i=0$, is in a non equilibrium configuration $u_{xt_i}=u^0_x$, whose statistical weight is given by $e^{-H_0[u^0]}$ (where $H_0[u_0]\neq H_{\rm CO}[u_0]$). The general framework to incorporate this feature in the MSR formalism has been developed in Ref. [42], and it amounts to describe the system in terms of the generating functional $S[u,i\hat{u}]\to S[u,i\hat{u}]+H_0[u^0_x]$. If the system is prepared in a high temperature state, with short range correlations $\langle u^0_x u^0_{x'}\rangle=m_0^{-2}\delta^d(x-x')$, the corresponding $H_0[u^0]$ is given by

$$H_0[u^0]=\frac{m_0^2}{2}\int_x(u^0_x)^2.\tag{121}$$

Any addition of anharmonic terms in $H_0[u^0]$ is irrelevant as long as $m_0^2\neq 0$. Moreover, by power counting one has that m_0^{-2} is irrelevant [42], so that to study the leading scaling behavior it is sufficient to assume $m_0^{-2}=0$, i.e., $u^0_x=0$. The effect of this nonequilibrium initial condition is then completely encoded in the lower bound $t_i=0$ on the time integrals in the MSR functional (72). The running bare response and correlation functions are given by [42]

$$R^q_{l_{tt'}}=\theta(t-t')e^{-q^2(t-t')}\left[c\left(\frac{q^2}{2\Lambda_0^2}\right)-c\left(\frac{q^2}{2\Lambda_l^2}\right)\right],\tag{122}$$

$$C^q_{l_{tt'}}=\frac{T}{q^2}\left[e^{-q^2|t-t'|}-e^{-q^2(t+t')}\right]\left[c\left(\frac{q^2}{2\Lambda_0^2}\right)-c\left(\frac{q^2}{2\Lambda_l^2}\right)\right].$$

B. Nonequilibrium response function

In order to compute the response function, we solve perturbatively the equation for $\mathcal{R}^{q}_{tt'}$ (99) using the trick explained above, i.e., replacing the exact $\mathcal{R}^{q}_{tt'}$ in the rhs of Eq. (99) by its bare value. Doing this, we obtain a perturbative expansion of the exponents z [already obtained previously (98)], θ, and of the scaling function $F_R(v,u)$ in the same spirit as Eq. (104). Indeed, as shown in Appendix F, one has the scaling (114), in terms of the scaling variables $v=\tilde{q}^z(\tilde{t}-\tilde{t}')$ and $u=\tilde{t}/\tilde{t}'$ with

$$F_R(v,u)=F_R^0(v)+\tau F_R^1(v,u), \tag{123}$$

$$F_R^1(v,u)=F_R^{1\mathrm{eq}}(v)+F_R^{1\mathrm{noneq}}(v,u),$$

$$\theta=e^{\gamma_E}\tau+O(\tau^2),$$

which is established by comparison with the direct perturbative expansion of Eq. (114) in powers of τ,

$$\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=F_R^0(v')+(z-2)\ln\tilde{q}[F_R^0(v')+v'F_R^{0'}(v')] + \theta\ln u F_R^0(v')+\tau F_R^1(v',u)+O(\tau^2) \tag{124}$$

with $v'=\tilde{q}^2(\tilde{t}-\tilde{t}')$ and $F_R^{1\mathrm{eq}}(v)$ is given by Eq. (104) and $F_R^{1\mathrm{noneq}}(v,u)$ given in Eq. (F16) has a complicated expression left in Appendix F (F18). However, its asymptotic behaviors, which we now focus on, have remarkably simple forms. First, in order to compare with the prediction for pure critical systems one is interested in the limit of large u, keeping v fixed. This defines $F_{R,\infty}(v)$ (116) which, we find to be

$$F_{R,\infty}(v)=e^{-v}+e^{\gamma_E}\tau\left\{-\sqrt{\pi v}\,\mathrm{erf}\sqrt{v}-e^{-v}\left[(1-v)\ln(4ve^{\gamma_E}) -2v\left(v-\frac{1}{2}\right)\ {}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},v\right)\right]\right\}+O(\tau^2), \tag{125}$$

where $\mathrm{erf}(z)$ is the error function and ${}_2F_2(\{1,1\},\{\frac{3}{2},2\},z)$ is a generalized hypergeometric series [54–56]. This shows that the response function has a scaling behavior as predicted for pure systems at a critical point (116). The small v behavior of $F_{R,\infty}(v)\sim 1-e^{\gamma_E}\tau\ln v$ shows that $\phi_R(v)$ (120) has a good limit when $v\to 0$, $\phi_R(0)=\mathrm{const}$, and this gives the autocorrelation exponent λ_R (119),

$$\lambda_R=2+O(\tau^2). \tag{126}$$

It is also interesting to analyze the asymptotic behavior in the limit of large v (and, in particular. $v\gg u$), keeping u fixed. This limit is relevant, e.g., to study the behavior at fixed q, large t,t' with $u=\tilde{t}/\tilde{t}'$ fixed. It is obtained from Eq. (F16) as explained in Appendix F. The behavior of the response function in this limit is then given by

$$\lim_{v\to\infty,u\,\mathrm{fixed}} F_R(v,u)\sim e^{-v}+\frac{\tau}{v^2}P_R(u)+O(\tau v^{-3},\tau^2), \tag{127}$$

$$P_R(u)=e^{\gamma_E}\frac{u+1}{2\sqrt{u}}.$$

Notice that in the limit $u\to 1$ we recover the result of the equilibrium dynamics (107). This is more general as one can check from Eq. (F18) that $F_R^{\mathrm{noneq}}(v,u)=O((u-1)^2)$ as $u\to 1$. Finally, one must keep in mind that the limits $v\to\infty$ and $u\to\infty$ do not commute, indeed one expects that a scaling function of $v/u\sim q^z t'$ interpolates between these limits, left for future investigation.

Another interesting behavior is the limit of vanishing momentum $\tilde{q}=0$, the so called diffusion mode. Although well defined, this limit is a bit peculiar due to the prefactor $\tilde{q}^{z-2}$ in the scaling function (114). However, the function $F_R(v,u)$ behaves when $v\to 0$ in such a way to cancel this divergence as in Eq. (106) and leads to a well defined response function $\mathcal{R}^{\tilde{q}=0}_{\tilde{t}\tilde{t}'}$ which has the scaling form

$$\mathcal{R}^{\tilde{q}=0}_{\tilde{t}\tilde{t}'}=\frac{1}{(\tilde{t}-\tilde{t}')^{(z-2)/z}}\left(\frac{\tilde{t}}{\tilde{t}'}\right)^{\theta}F_R^{\mathrm{diff}}\left(\frac{\tilde{t}}{\tilde{t}'}\right), \tag{128}$$

$$F_R^{\mathrm{diff}}(u)=F_R^{\mathrm{diff0}}(u)+\tau F_R^{\mathrm{diff1}}(u),$$

$$F_R^{\mathrm{diff0}}(u)=1,$$

$$F_R^{\mathrm{diff1}}(u)=2e^{\gamma_E}\ln\left(\frac{1+\sqrt{u}}{2\sqrt{u}}\right),$$

which is identified with the perturbative expansion of $\mathcal{R}^{\tilde{q}=0}_{\tilde{t}\tilde{t}'}$ straighforwardly obtained from the general expression (F18).

C. Nonequilibrium correlation function

To compute the correlation function, instead of solving the equation for $\mathcal{C}^{q}_{ltt'}$ (100), we obtain it using the following formal solution for $\tilde{t}>\tilde{t}'$:

$$\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\lim_{l\to\infty}\mathcal{C}^{\tilde{q}}_{l\tilde{t}\tilde{t}'} =2T\int_0^{\tilde{t}'}dt_1\mathcal{R}^{\tilde{q}}_{\tilde{t}t_1}\mathcal{R}^{\tilde{q}}_{\tilde{t}'t_1}+\int_0^{\tilde{t}}dt_1\int_0^{\tilde{t}'}dt_2\mathcal{R}^{\tilde{q}}_{\tilde{t}t_1}D_{t_1t_2}\mathcal{R}^{\tilde{q}}_{\tilde{t}'t_2}, \tag{129}$$

where $D_{t_1t_2}=\lim_{l\to\infty}D_{lt_1t_2}$ is defined in Eq. (88) and explicitly given in Eq. (F4), which we expand perturbatively using the expression we obtained for $\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ (124). In Appendix F, we show that $\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ has the following scaling form (115) with

$$F_C(v,u)=F_C^0(v,u)+\tau F_C^1(v,u),$$

$$F_C^0(v,u)=e^{-v}-e^{-v(1+u)/(u-1)}, \tag{130}$$

and $F_C^1(v,u)$ given in Appendix F. Again, this is established by identifying the direct perturbative expansion of Eq. (115):

$$\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\frac{T}{\tilde{q}^2}\left[F_C^0(v',u)+(z-2)\ln(\tilde{q})v'\frac{\partial F_C^0(v',u)}{\partial v'}+\theta\ln u F_C^0(v',u)+\tau F_C^1(v',u)\right] \tag{131}$$

with $v'=\tilde{q}^2(\tilde{t}-\tilde{t}')$, which is similar to the scaling form expected for pure systems at a critical point (115). However, the large u behavior is *different*, indeed one has in the large u limit, keeping v fixed

$$\lim_{u\to\infty}F_C(v,u)\sim\frac{2e^{-v}v}{u}+\tau\frac{F^1_{C,\infty}(v)}{\sqrt{u}}+O(u^{-2},\tau u^{-1},\tau^2), \tag{132}$$

$$F^1_{C,\infty}(v)=e^{\gamma_E}e^{-v}\sqrt{\pi v}\,\mathrm{erfi}\sqrt{v},$$

which decays more slowly than the predicted scaling for pure system at a critical point (117). Besides, using Eq. (F23) $F^1_{C,\infty}(v)\sim v+O(v^2)$, $\phi_C(0)=\text{const}$ (120), this defines the autocorrelation exponent λ_C,

$$\lambda_C=d-\frac{z}{2}+O(\tau^2)=1-e^{\gamma_E}\tau+O(\tau^2), \tag{133}$$

where we have used the explicit expressions of the exponents z (98), θ (123), and the relation $(d-\lambda_c)z^{-1}=\theta-1/2+(2-\eta)z^{-1}$ arising from $1/\sqrt{u}$ decay of $F_C^1(v,u)$ (132). Note first that λ_C is different from its trivial value $\lambda_C\neq d$. Besides, we note that $\lambda_C\neq\lambda_R$ and finally that it violates the bound $\lambda_C<d/2$ predicted for pure systems.

For the correlation function it is also instructive to look at the asymptotic behavior $v\gg1$, u fixed. As detailed in Appendix F, one has

$$\lim_{v\to\infty}F_C(v,u)\sim F_C^0(v,u)+\frac{\tau}{v}P_C(u)+O(\tau v^{-2},\tau^2), \tag{134}$$

$$P_C(u)=P_R(u).$$

Finally, we also study the correlation function in the limit of vanishing momentum $\tilde{q}=0$. As mentioned previously for the response function, this limit is a bit peculiar due to the q^{-2} prefactor in Eq. (115). The small v behavior of $F_C(v,u)$ leads to the scaling form (up to a nonuniversal scale)

$$\mathcal{C}^{\tilde{q}=0}_{\tilde{t}\tilde{t}'}=2\tilde{t}'T\frac{1}{(\tilde{t}-\tilde{t}')^{(z-2)/z}}\left(\frac{\tilde{t}}{\tilde{t}'}\right)^{\theta}F_C^{\mathrm{diff}}(u), \tag{135}$$

$$F_C^{\mathrm{diff}}(u)=F_C^{\mathrm{diff}0}(u)+\tau F_C^{\mathrm{diff}1}(u),$$

$$F_C^{\mathrm{diff}0}(u)=1,$$

$$F_C^{\mathrm{diff}1}(u)=\frac{1}{2}e^{\gamma_E}[4\sqrt{u}+(u+1)\ln(u-1)-2(u-1)\ln(1+\sqrt{u})-2\ln u+6-8\ln 4],$$

with the asymptotic behaviors

$$F_C^{\mathrm{diff}}(u)\sim1+\tau e^{\gamma_E}\ln(u-1),\quad u\to1^+, \tag{136}$$

$$F_C^{\mathrm{diff}}(u)\sim1+\tau e^{\gamma_E}\sqrt{u},\quad u\gg1.$$

These behaviors are such that the singularity as $\tilde{t}-\tilde{t}'\to0$ cancels and one finds that the diffusion of the zero mode become anomalous at large time,

$$\mathcal{C}^{\tilde{q}=0}_{\tilde{t}\tilde{t}'}\sim A\tilde{t}'^{2/z}, \tag{137}$$

$$A=2T_c+O(\tau),$$

this formula being valid for $\tilde{t}-\tilde{t}'\ll\tilde{t}'$, the random walk result being recovered when $z=2$.

D. Fluctuation dissipation ratio

We now give the results for the FDR $X^q_{tt'}$ defined by [47]

$$\frac{T}{X^q_{tt'}}=\frac{\partial_{t'}\mathcal{C}^q_{tt'}}{\mathcal{R}^q_{tt'}}. \tag{138}$$

Starting from the scaling laws that we established above, we can compute the FDR $X^q_{tt'}\equiv X^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ as a function of the scaling variables $\tilde{q}^z(\tilde{t}-\tilde{t}')$ and $\tilde{t}/\tilde{t}'$. As we saw previously, both the exponent z and the scaling function associated with the FDR will have an expansion in powers of τ, i.e.,

$$\frac{T}{X^{\tilde{q}}_{\tilde{t}\tilde{t}'}}=F_X\left(\tilde{q}^z(\tilde{t}-\tilde{t}'),\frac{\tilde{t}}{\tilde{t}'}\right), \tag{139}$$

$$F_X(v,u)=F_X^0(v,u)+\tau F_X^1(v,u)+O(\tau^2),$$

the expansion of z to order τ being given by Eq. (98). $F_X^0(v,u)$ corresponds to the Gaussian model and from the perturbative expansions that we obtained for $\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ (124) and $\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ (131) one can identify (perturbatively) this scaling form, with

$$F_X^0(v,u)=1+e^{-2v/(u-1)}, \tag{140}$$

$$\begin{aligned}F_X^1(v,u)=&-\frac{\theta}{\tau}\frac{u-1}{v}(1-e^{-2v/(u-1)})\\&-e^v F_R^1(1+e^{-2v/(u-1)})\\&-e^v\left(\frac{\partial F_C^1(v,u)}{\partial v}+\frac{u(u-1)}{v}\frac{\partial F_C^1(v,u)}{\partial u}\right).\end{aligned}$$

Inserting the formulas for F_R^1 and F_C^1 obtained in Appendix F yields the general result for F_X as a nontrivial function of the two variables u,v. Here we only give the behaviors of this scaling function in the different asymptotic limits studied previously. First, we note that this formula gives back the FDT result $F_X=1$ for $u=1$.

Second, focusing on the limit $u\gg1$, keeping v fixed, one has

$$\begin{aligned}F_X(v,u)=&1+e^{-2v/(u-1)}+\frac{\sqrt{\pi}}{2}e^{\gamma_E}\tau\sqrt{\frac{u}{v}}\,\mathrm{erfi}\sqrt{v}\\&+O(\tau u^0,\tau^2).\end{aligned} \tag{141}$$

Thus in this regime X decreases below its FDT value $X_{FDT}=1$. Looking at this result, one is tempted to conclude that $X_{tt'}^q$ vanishes as $t/t'\to\infty$ when $q^z(t-t')$ is kept fixed. In particular, for $q=0$ (see below the direct calculation in this case) one finds the analogous quantity $X_\infty^{q=0}$ computed in Refs. [45,46] for several models. However, one must keep in mind that Eq. (141) is perturbative in τ and the divergence of the coefficient of τ could also be a sign of a nonanalyticity in τ of the $u=\infty$ result. Elucidation of this point is left for future study.

In the other limit that we studied previously, corresponding to $v\gg1$, keeping u fixed, we obtain straightforwardly the following behavior:

$$\begin{aligned}F_X(v,u)=&1+e^{-2v/(u-1)}-\frac{e^{\gamma_E}\tau e^v}{2v^2}\frac{(u-1)^2}{2\sqrt{u}}\\&+O(\tau e^v v^{-3},\tau^2).\end{aligned} \tag{142}$$

This limit is relevant to study fixed q. It shows that there is still aging behavior in a given nonzero mode, and appears to contradict some claims [45] that only the zero mode (diffusion) exhibits interesting aging behavior. Note also that in this regime one has $X>X_{FDT}$, a feature found in other disordered models [57].

Finally, in the limit of vanishing momentum $\tilde{q}=0$, the FDR is a function of the scaling variable $\tilde{t}/\tilde{t}'$ whose perturbative expansion is given by

$$\frac{T}{X_{\tilde{t}\tilde{t}'}^{\tilde{q}=0}}=F_X^{\mathrm{diff}}\left(\frac{\tilde{t}}{\tilde{t}'}\right), \tag{143}$$

$$F_X^{\mathrm{diff}}(u)=F_X^{\mathrm{diff}0}(u)+\tau F_X^{\mathrm{diff}1}(u)+O(\tau^2), \tag{144}$$

$$F_X^{\mathrm{diff}0}(u)=2,$$

$$\begin{aligned}F_X^{\mathrm{diff}1}(u)=&2F_C^{\mathrm{diff}1}(u)-2u\frac{dF_C^{\mathrm{diff}1}(u)}{du}-2F_R^{\mathrm{diff}1}(u)\\&+\frac{2(z-2)}{z\tau(u-1)}-2\frac{\theta}{\tau}.\end{aligned}$$

Using the results of previous sections, we find

$$\frac{T}{X_{\tilde{t}\tilde{t}'}^{\tilde{q}=0}}=2+\tau e^{\gamma_E}\left[\sqrt{u}+\ln\left(\frac{\sqrt{u}-1}{\sqrt{u}+1}\right)+\sigma\right], \tag{145}$$

where σ is a numerical constant. This constant depends on additive constants to, respectively, F_R^1 and F_C^1, each of them being nonuniversal as discussed above (see Appendix). However, a distinct possibility is that $F_X^{\mathrm{diff}}(u)$ is universal (i.e., that the nonuniversal parts cancel). Checking this can be done with the present method, and is left for future study. The value obtained here, $\sigma=5-12\ln 2$, may only be indicative since we did not keep track of all additive constants. In particular, in the scaling regime $\tilde{t}\gg\tilde{t}'\gg1$, one obtains

$$\frac{T}{X_{\tilde{t}\tilde{t}'}^{\tilde{q}=0}}\sim2+\tau e^{\gamma_E}\sqrt{u}+O(\tau u^0,\tau^2). \tag{146}$$

Notice that taking the limit $v\to0$ [using Eq. (F23)] on the asymptotic expression (141) where we have taken the limit $u\gg1$ *before* v small, one recovers the same result (146).

One way to understand the result (145), i.e., the divergence of $X_{tt'}^{\tilde{q}=0}$ when $t'\to t$ is to note that the same divergence occurs for a simple diffusion process with the same close times asymptotic behaviors:

$$\mathcal{C}_{tt'}^{\tilde{q}=0}\sim t'^{2/z}, \tag{147}$$

$$\mathcal{R}_{tt'}^{\tilde{q}=0}\sim(t-t')^{(2-z)/z}, \tag{148}$$

which yields straightforwardly $X_{tt'}^{\tilde{q}=0}\sim A(u-1)^{(2-z)/z}$ as $u\to1$. Note, however, that to obtain the correct *amplitude* A one needs to take into account further corrections to $\mathcal{C}_{tt'}^{\tilde{q}=0}$, specifically we note that one can rewrite Eq. (135) as

$$\mathcal{C}_{tt'}^{\tilde{q}=0}=t'^{2/z}\mathcal{A}(u) \tag{149}$$

and that the detailed asymptotics of $\mathcal{A}(u)$ near $u=1$ determines the amplitude of the divergence.

VII. CONCLUSION

In this paper we have developed an EMRG method to perform first principle perturbative calculations based on exact RG. Contrarily to previous works, it is based on a multilocal expansion of the effective action functional. It allows us to conveniently perform calculations with an arbitrary cutoff function in a fully controlled way and to check explicitly the universality of the observables.

We have tested the method on the standard $O(N)$ model. We have shown that the exponent η to order $O(\epsilon^2)$ can be simply recovered within the exact RG multilocal expansion. This is interesting since previous approaches relied on approximations such as polynomial and derivative expansions, which are not needed here. We have also obtained several two-point scaling functions and explicitly checked universality. Finally, we explained how the method compares with more standard field theoretical approaches. In a sense, the present method directly yields the renormalized theory.

We have applied the EMRG method to study the glass phase of the two dimensional random sine Gordon model (Cardy-Ostlund) near the glass transition temperature. We have first recovered known results for the statics and for the equilibrium dynamical exponent z which we showed to be universal. The method of derivation, however, is quite different from previous ones, since it yields directly the self-energy $\Sigma_l(\omega)$ as a scaling function of ω/Λ_l, where Λ_l is the infrared cutoff. We have given the scaling functions associated with finite momentum equilibrium response and correlation.

Next we studied the out of equilibrium dynamics of the Cardy-Ostlund model. We obtained the two time response and correlations at finite momentum. These were found to take a scaling form and we computed analytically the corresponding scaling functions which depend on two arguments $v=\tilde{q}^z(\tilde{t}-\tilde{t}')$ and $u=\tilde{t}/\tilde{t}'$. We showed that they exhibit aging behavior characterized by a nontrivial fluctuation dissipation ratio X, itself a universal function of u,v that we obtained. We also obtained the off equilibrium exponents θ and λ. Interestingly we found that, at variance with pure systems, one must introduce two distinct exponents λ_R and λ_C for response and correlation, respectively. Our study raises the question of whether this could be a more general property of glassy dynamics in disordered systems.

Our method is promising for further RG studies of disordered systems, as it allows to attack the problem with few assumptions. Other situations where it can be applied are elastic manifolds in random media, where it can be used to put the so-called functional RG on a more solid basis [58,59]. Concerning the results of the present paper, a numerical simulation of the Cardy-Ostlund glass phase can be performed [60] and should provide an interesting test of the predictions of our RG calculation. In particular, some points require further examination, e.g., the asymptotic value X_∞ of the FDR. This would be interesting especially in the light of the present activity on FDR in mean field models, and interpretations in terms of effective temperatures. Indeed, developing real space, RG type methods beyond mean field remains a challenge in the theory of glasses.

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APPENDIX A: EXACT RG EQUATION FOR THE EFFECTIVE ACTION

Here we present a simple derivation of the exact RG equation satisfied by the effective action, denoted here $\Gamma_G(\phi)$ [and $\Gamma(\phi)$ in the text], for the theory of action given in Eq. (1), when the propagator G is varied, for a fixed interacting functional $\mathcal{V}(\phi)$. One first introduces the generating functional

$$Z_G(j)=\int D\phi\; e^{-1/2\phi:G^{-1}:\phi-\mathcal{V}(\phi)+j:\phi}, \tag{A1}$$

i.e., the partition function in presence of a set of sources denoted $j\equiv j_x^i$. For any variation ∂G of G, its variation $\partial Z_G(j)$ satisfies

$$\begin{aligned}\partial Z_G(j)&=-\frac{1}{2}\mathrm{Tr}\partial G^{-1}\int D\phi\,\phi\phi e^{-1/2\phi:G^{-1}:\phi+\mathcal{V}(\phi)+j:\phi}\\&=-\frac{1}{2}\mathrm{Tr}\partial G^{-1}\frac{\delta^2 Z_G(j)}{\delta j\,\delta j},\end{aligned} \tag{A2}$$

where $\partial G^{-1}=-G^{-1}\partial G G^{-1}$ and Tr denotes a trace over all spatial coordinates and indices x,i. Next, one introduces the generating functional $W_G(j)=\ln Z_G(j)$ of connected correlations, which varies as

$$\partial W_G(j)=-\frac{1}{2}\mathrm{Tr}\partial G^{-1}\left(\frac{\delta^2 W_G(j)}{\delta j\,\delta j}+\frac{\delta W_G(j)}{\delta j}\frac{\delta W_G(j)}{\delta j}\right), \tag{A3}$$

an exact RG equation for this quantity. From there it is simple to obtain the RG equation obeyed by its Legendre transform $\Gamma_G(\phi)=\min_j[\phi:j-W_G(j)]$. We will assume that no problem arises from the convexity condition and that $\Gamma_G(\phi)$ can be obtained using only the saddle point conditions,

$$\frac{\delta W_G}{\delta j}[j_G(\phi)]=\phi, \tag{A4}$$

$$\frac{\delta \Gamma_G(\phi)}{\delta \phi}=j_G(\phi). \tag{A5}$$

For the variation of $\Gamma_G(\phi)=\phi:j_G(\phi)-W_G(j_G(\phi))$, this yields

$$\begin{aligned}\partial\Gamma_G(\phi)&=-\partial W_G[j_G(\phi)]\\&=\frac{1}{2}\mathrm{Tr}\partial G^{-1}\left(\frac{\delta^2 W_G}{\delta j\,\delta j}[j_G(\phi)]\right.\\&\quad\left.+\frac{\delta W_G}{\delta j}[j_G(\phi)]\frac{\delta W_G}{\delta j}[j_G(\phi)]\right)\end{aligned} \tag{A6}$$

since the term proportional to $\partial_G j_G$ cancels as usual from the saddle point conditions (A5). Using Eq. (A5) once more, as well as the standard relation $\delta W_G/\delta j\,\delta j[j_G(\phi)]=[\delta\Gamma_G/\delta\phi\,\delta\phi]^{-1}$, gives Eq. (5) of the text.

Writing then

$$\Gamma_G(\phi)=\frac{1}{2}\phi:G^{-1}:\phi+\mathcal{U}_G(\phi)-\frac{1}{2}\mathrm{Tr}\ln G, \tag{A7}$$

this is equivalent to the equation for $\mathcal{U}_G$,

$$\partial\mathcal{U}_G(\phi)=\frac{1}{2}\mathrm{Tr}\,\partial G:\left[G^{-1}-G^{-1}\left(1+G:\frac{\delta^2\mathcal{U}_G}{\delta\phi\delta\phi}\right)^{-1}\right] \tag{A8}$$

or its equivalent form given in the text.

Now that we have an exact equation for $\Gamma_G(\phi)$, we can relate the effective action in theories with the same $\mathcal{V}(\phi)$ but different G. All we need to fully determine the effective action is an initial condition. It is provided by the action itself. Indeed, one has the following perturbative loop expansion:

$$\Gamma(\phi)=-\frac{1}{2}\mathrm{Tr}\ln G+\mathcal{S}(\phi)+\sum_{k\geq 1}\Gamma^k(\phi), \tag{A9}$$

where $\Gamma^k(\phi)$ is the sum of all k loop 1PI graphs using $\mathcal{V}(\phi)$ as interaction and G as propagator. Thus, if the initial condition for the propagator $G_{l=0}$ is such that all $\Gamma^k(\phi)$ graphs vanish when computed with $G_{l=0}$, then one can choose the initial condition as $\mathcal{U}_{l=0}(\phi)=\mathcal{V}(\phi)$. This is the case for the choice (6), (8) made in the text [similarly the initial condition for $W_G(j)$ in Eq. (A3) is the Legendre transform of the initial action $\mathcal{S}(\phi)$].

Finally, let us note that the RG equation can also be written as

$$\frac{d\mathcal{U}_G(\phi)}{dG}=\frac{\partial}{\partial G}\frac{1}{2}\mathrm{Tr}\ln\left(1+G:\frac{\delta^2\mathcal{U}_G(\phi)}{\delta\phi\delta\phi}\right) \tag{A10}$$

$$=\frac{1}{2}\frac{\delta^2\mathcal{U}_G(\phi)}{\delta\phi\delta\phi}:\left(1+G:\frac{\delta^2\mathcal{U}_G(\phi)}{\delta\phi\delta\phi}\right)^{-1}, \tag{A11}$$

where the derivative $\partial/\partial G$ in the rhs of the first equation is restricted to the *explicit* G dependence [i.e., not the one implicit in $\mathcal{U}_G(\phi)$].

APPENDIX B: MULTILOCAL EXPANSION TO $O(U^2)$

To $O(U^2)$ one needs only U and $V\sim O(U^2)$ in the expansion (10) of $\tilde{U}$. The functional derivative reads

$$\frac{\delta\tilde{U}}{\delta\phi_x^i\delta\phi_y^j}=\delta_{xy}\left[\partial_i\partial_j U(\phi_x)+\int_z\partial_i^1\partial_j^1 V(\phi_x,\phi_z,x-z)+\partial_i^2\partial_j^2 V(\phi_z,\phi_x,z-x)\right]+2\,\partial_i^1\partial_j^2 V(\phi_x,\phi_y,x-y)+O(W), \tag{B1}$$

using parity $V(\phi,\psi,-x)=V(\phi,\psi,x)$. Inserting in Eq. (9) and keeping only terms up to order $O(U^2)$, one finds the resulting RG equation:

$$\partial_l U_l(\phi)=\frac{1}{2}\partial G_{ij}^{x=0}\partial_i\partial_j U_l(\phi)+\int_x\partial G_{ij}^x\partial_i^1\partial_j^2 V_l(\phi,\phi,x)-\frac{1}{2}\int_x\partial G_{ij}^x\partial_j\partial_k U_l(\phi)(G_l^x)_{km}\partial_m\partial_i U_l(\phi), \tag{B2}$$

$$\begin{aligned}\partial_l V_l(\phi,\psi,x)=&-\frac{1}{2}\partial G_{ij}^x\partial_j\partial_k U_l(\phi)(G_l^x)_{km}\partial_m\partial_i U_l(\psi)\\&+\frac{1}{2}\partial G_{ij}^{x=0}(\partial_i^1\partial_j^1+\partial_i^2\partial_j^2)V_l(\phi,\psi,x)\\&+\partial_i^1\partial_j^2\left(\partial G_{ij}^x V_l(\phi,\psi,x)-\delta(x)\int_y\partial G_{ij}^y V_l(\phi,\psi,y)\right)\\&+\frac{1}{2}\delta(x)\int_y\partial G_{ij}^y\partial_j\partial_k U_l(\phi)\\&\times(G_l^y)_{km}\partial_m\partial_i U_l(\psi),\end{aligned} \tag{B3}$$

where the local projection $\overline{P_1}$ operator has been applied to obtain the first equation, and the operator $1-P_1$ to obtain the second. This is illustrated in Fig. 2 [dropping all terms of order $O(U^3)$ and higher]. Note that $\int_x V(\phi,\psi,x)=0$. The differential equation for the bilocal part V_l is linear, a general property that allows to solve all higher multilocal components (here V_l) as a function of the local part U_l only. The equation for V_l can be integrated in the forms (12), (13) given in the text. The method is similar to Ref. [1] to which we refer for further details. Inserting this solution in the equation for U_l, one obtains Eq. (14) in the text. We have assumed that no bilocal term exists in the original action. Near the fixed point form at large l these assumptions are not strictly necessary, a statement that can be checked using the present method.

It can be useful, in particular for the Cardy-Ostlund model, that we study in the text, to introduce a Fourier representation in the fields,

$$U_l^K=\int d\phi e^{-iK\cdot\phi}U_l(\phi),$$

$$V_l^{K,Px}=\int d\phi d\psi e^{-iK\cdot\phi-iP\cdot\psi}V_l(\phi,\psi,x). \tag{B4}$$

Using this representation, we obtain the RG equations (12)–(14) in Fourier space,

$$V_l^{K,Px}=\frac{1}{2}\left(F_l^{K,Px}-\delta(x)\int_y F_l^{K,Py}\right), \tag{B5}$$

$$F_l^{K,Px} = -\int_0^l dl' (K\cdot\partial G_{l'}^x \cdot P)(K\cdot G_{l'}^x\cdot P) \times e^{1/2K.G_{l'l}^{x=0}\cdot K + \frac{1}{2}P\cdot G_{l'l}^{x=0}\cdot P + K\cdot G_{l'l}^x \cdot P} U_{l'}^K U_{l'}^P$$

$$\partial_l U_l^K = \frac{-1}{2} K\cdot\partial G^{x=0}\cdot K U_l^K - \frac{1}{2}\int_{P,Q,P+Q=K}\int_x (P\cdot\partial G^x\cdot Q) \times (P\cdot G_l^x\cdot Q) U_l^P U_l^Q + \frac{1}{2}\int_{P,Q,P+Q=K}\int_x P\cdot(\partial G_l^x - \partial G_l^0)\cdot Q\int_0^l dl'(P\cdot\partial G_{l'}^x\cdot Q)(P\cdot G_{l'}^x\cdot Q) \times e^{1/2P\cdot G_{l'l}^{x=0}\cdot P + \frac{1}{2}Q\cdot G_{l'l}^{x=0}Q + P\cdot G_{l'l}^x\cdot Q} U_{l'}^P U_{l'}^Q, \tag{B6}$$

where $\int_{P,Q,P+Q=K} \equiv \int [d^N P d^N Q/(2\pi)^N]\delta(K-P-Q)$ where N is the number of components of ϕ. In the text we have used $\hat{V}_l^{K,Px}$ to distinguish the Fourier series coefficients from the Fourier transform.

APPENDIX C: DETAILED CALCULATIONS FOR THE $O(N)$ MODEL

1. β function

Let us insert Eq. (18) into the ERG equation (14), keeping only g_0, g_2, and g_4 for now, and first focus on the first line in Eq. (14), which reads

$$\partial_l\left[g_{0,l} + \frac{\tilde{g}_{2,l}}{2!}\Lambda_l^2\phi^2 + \frac{\tilde{g}_{4,l}}{4!}\Lambda_l^\epsilon(\phi^2)^2\right] = \frac{1}{2}\int_q \partial G_l^q \partial_i\partial_i U_l(\phi) - \frac{1}{2}\int_q \partial G_l^q G_l^q[\partial_i\partial_j U_l(\phi)]^2 \tag{C1}$$

with implicit sums on repeated indices. Using that

$$\partial_i\partial_j U_l(\phi) = g_{2,l}\delta_{ij} + \frac{g_{4,l}}{3!}(\delta_{ij}\phi^2 + 2\phi^i\phi^j), \tag{C2}$$

which yields

$$\partial_i\partial_i U_l = N g_{2,l} + \frac{N+2}{3!}g_{4,l}\phi^2, \tag{C3}$$

$$\partial_i^1\partial_j^1 U_l(\phi_1)\partial_i^2\partial_j^2 U_l(\phi_2) = N g_{2,l}^2 + \frac{N+2}{3!}g_{2,l}g_{4,l}(\phi_1^2+\phi_2^2) + \frac{1}{(3!)^2}g_{4,l}^2[(N+4)\phi_1^2\phi_2^2 + 4(\phi_1\cdot\phi_2)^2].$$

Setting $\phi_1=\phi_2$ in Eq. (C3) and identifying the coefficients of ϕ^2 and $(\phi^2)^2$, one then easily obtains all terms in Eqs. (21) and (20) apart from the last one, with the scaled integrals defined in Eq. (22).

Inserting now Eq. (18) into the second line of the ERG equation (14), one obtains only a correction to g_2 from the term with the lowest number of derivatives (six). Noting that

$$\partial_i\partial_j\partial_k U_l(\phi) = \frac{g_{4,l}}{3}(\delta_{ij}\phi^k + \delta_{ik}\phi^j + \delta_{jk}\phi^i), \tag{C4}$$

one finds

$$\left(\sum_i \partial_i^1\partial_i^2\right)^3 U_l(\phi_1)U_l(\phi_2) = \frac{g_{4,l}^2}{3}(N+2)\phi_1\cdot\phi_2 \tag{C5}$$

yielding the last term in Eq. (21),

$$-\frac{N+2}{3}\int_0^l dl'\tilde{I}_{l,l'}^{(2)}\tilde{g}_{4,l'}^2, \tag{C6}$$

with

$$\tilde{I}_{l,l'}^{(2)} = \Lambda_l^{-2}\int_x(\partial_l G_l^x - \partial_l G_l^{x=0})\partial_{l'}G_{l'}^x G_{l'}^x \Lambda_{l'}^{2\epsilon}. \tag{C7}$$

This term does not modify the fixed point value $\tilde{g}_2^*$ to order ϵ, provided it remains finite in the limit $l\to\infty$. A way to compute it is to make an integration by part to treat the integral over l',

$$\int_0^l dl'\partial_{l'}G_{l'}^x G_{l'}^x\tilde{g}_{4,l'}^2\Lambda_{l'}^{2\epsilon} = \frac{1}{2}(G_l^x)^2\tilde{g}_{4,l}^2\Lambda_l^{2\epsilon} - \int_0^l dl'(G_{l'}^x)^2\tilde{g}_{4,l'}\Lambda_{l'}^\epsilon\partial_{l'}(\tilde{g}_{4,l'}\Lambda_{l'}^\epsilon) = \frac{1}{2}(G_l^x)^2\tilde{g}_{4,l}^2 + O(\epsilon\tilde{g}_{4,l}^2, \tilde{g}_{4,l}^3) \tag{C8}$$

as from Eq. (20) $\partial_{l'}(\tilde{g}_{4,l'}\Lambda_{l'}^\epsilon)$ is of order $\tilde{g}_{4,l'}^2$ and where we have used $G_{l=0}^x = 0^+$. The terms we dropped are of order ϵ^3 in the limit $l\to\infty$. Finally, in the large l limit we are left with

$$\int_0^l dl'\tilde{I}_{l,l'}^{(2)}\tilde{g}_{4,l'}^2 = \frac{\tilde{g}_{4,l}^2}{2}\Lambda_l^{-2}\int_x(\partial_l G_l^x - \partial_l G_l^{x=0})(G_l^x)^2 + O(\epsilon^3), \tag{C9}$$

which is already of order ϵ^2, so that the integral over x can be performed in $d=4$ exactly. Using the decomposition of the cutoff (17), we compute the following integrals exactly in $d=4$:

$$G_l^x = \frac{1}{4\pi^2}\int_a \frac{1}{x^2}(e^{-x^2\Lambda_l^2/2a} - e^{-x^2\Lambda_0^2/2a}),$$

$$\Lambda_l^{-2}\partial_l G_l^x = \frac{1}{4\pi^2}\int_a \frac{1}{a}e^{-x^2\Lambda_l^2/2a}. \tag{C10}$$

Equation (C6) can finally be written as an integral over the rescaled variable $\tilde{x}=\Lambda_l x$,

$$\frac{N+2}{3}\int_0^l dl' \tilde{I}^{(2)}_{l,l'}\tilde{g}^2_{4,l'} \propto \tilde{g}^2_{4,l}\int_{\tilde{x}}\int_a (e^{-\tilde{x}^2/2a}-1)\frac{1}{a\tilde{x}^4} \times\left(\int_a e^{-(\Lambda_0/\Lambda_l)^2\tilde{x}^2/(2a)}-e^{-\tilde{x}^2/2a}\right)^2. \tag{C11}$$

In the limit $l\to\infty$, this integral is well defined. Indeed, there is no UV divergence [due to the term $(e^{-\tilde{x}^2/2a}-1)$ which behaves as $\tilde{x}^2$] nor IR divergence due to the term $e^{-\tilde{x}^2/a}$.

By the same calculation one obtains the flow of the free energy,

$$\partial_l g_{0,l}=\frac{N}{2}\Lambda_l^d(\tilde{I}^{(0)}_l\tilde{g}_{2,l}-\tilde{I}^{(1)}_l\tilde{g}^2_{2,l})+\frac{N+2}{3}\Lambda_l^d\int_0^l \tilde{I}^{(3)}_{l,l'}g^2_{4,l'}, \tag{C12}$$

with

$$\tilde{I}^{(3)}_{l,l'}=\Lambda_l^{-d}\int_x(\partial_l G^x_l-\partial_l G^{x=0}_l)\partial_{l'}G^x_{l'}G^x_{l'}G^x_{l'l}\Lambda^{2\epsilon}_{l'}.$$

We finally obtain the flow for $\tilde{g}_{6,l}$ in Eq. (18) with the same kind of manipulations, and using, furthermore,

$$\partial_i\partial_j((\phi^2)^3)=6\,\delta_{ij}(\phi^2)^2+24\phi^i\phi^j\phi^2,$$

$$\partial_i\partial_j\partial_k((\phi^2)^3)=24\phi^2(\delta_{ij}\phi^k+\delta_{ik}\phi^j+\delta_{jk}\phi^i)+48\phi^i\phi^j\phi^k, \tag{C13}$$

one gets

$$\partial_l\tilde{g}_{6,l}=(2\epsilon-2)\tilde{g}_{6,l}-(N+14)\tilde{I}^{(1)}_l\tilde{g}_{4,l}\tilde{g}_{6,l} -\frac{8}{5}(3N+16)\int_0^l dl'\tilde{I}^{(2)}_{l,l'}\tilde{g}^2_{6,l'}+O(\tilde{g}^3_{4,l}), \tag{C14}$$

which shows that $\tilde{g}^*_6\sim\epsilon^3$. Similarly, there is a term proportional to $\tilde{I}^{(0)}\tilde{g}_{6,l}$ in the flow equation of $\tilde{g}_{4,l}$ which affects the fixed point value $\tilde{g}^*_4$ only to next order in ϵ.

2. Computation of the exponent η

The quadratic term in Eq. (29) is obtained by inserting Eq. (18) in Eq. (13) and expanding the exponential in Eq. (13) to order 1. One gets, using Eq. (C5),

$$\begin{aligned}F_l(\phi_1,\phi_2,x)&=\int_0^l dl'\,\partial G^x_{l'}G^x_{l'}G^x_{l'l}\times\left(\sum_i \partial^1_i\partial^2_i\right)^3 U_l(\phi_1)U_l(\phi_2)\\&=\frac{N+2}{3}\phi_1\cdot\phi_2\int_0^l dl'\,\partial_{l'}G^x_{l'}G^x_{l'}G^x_{l'l}\tilde{g}^2_{4l'}\Lambda^{2\epsilon}_{l'},\end{aligned} \tag{C15}$$

which is the second line of Eq. (29). We have dropped terms of the form $f(\phi_i,x)$ such as $\tilde{g}_{2,l}\tilde{g}_{4,l}\phi_1^2$, $\tilde{g}_{2,l}\tilde{g}_{4,l}\phi_2^2$ [resulting from the expansion of the exponential in Eq. (13) to order 0], or $\tilde{g}^2_{4,l}\phi_1^2$, $\tilde{g}^2_{4,l}\phi_2^2$ [resulting from the expansion of the exponential in Eq. (13) to order 1 but acting, respectively, with $\partial^2\cdot G^{x=0}_{l'l}\cdot\partial^2$ or $\partial^1\cdot G^{x=0}_{l'l}\cdot\partial^1$] because they do not give any contribution to the effective action. Indeed, the contribution of such terms to the interaction functional $\mathcal{U}_l(\phi)$ (10), will be

$$V_l(\phi_x,\phi_y,x-y)=f(\phi_x,x-y)-\delta(x-y)\int_z f(\phi_x,z),$$

$$\begin{aligned}\mathcal{U}_l(\phi)&\sim\int_{x,y}\left(f(\phi_x,x-y)-\delta(x-y)\int_z f(\phi_x,z)\right)\\&=\int_{x,y}f(\phi_x,x-y)-\int_{x,z}f(\phi_x,z)=0,\end{aligned}$$

where we assumed parity $f(\phi_i,x)=f(\phi_i,-x)$ (which is the case here) and translational invariance. To treat the integral over l' in Eq. (C15), we use an integration by part as in Eq. (C8), one gets

$$\begin{aligned}&\frac{N+2}{3}\phi_1\cdot\phi_2\int_0^l dl'\,\partial_{l'}G^x_{l'}G^x_{l'}G^x_{l'l}\tilde{g}^2_{4l'}\Lambda^{2\epsilon}_{l'}\\&=-\frac{N+2}{18}\tilde{g}^2_{4,l}\phi_1\cdot\phi_2(G^x_l)^3+O(\epsilon\tilde{g}^2_{4,l},\tilde{g}^3_{4,l}),\end{aligned}$$

which leads to Eq. (30). Using Eq. (C10), the last term in Eq. (30) reads (forgetting for the discussion the numerical prefactor)

$$\begin{aligned}H(q,\Lambda_0,\Lambda_l)&=\int_x(e^{iqx}-1)(G^x_l)^3\\&=\frac{1}{(4\pi^2)^3}\int_x(e^{iqx}-1)\frac{1}{x^6}\times\left(\int_a e^{-x^2\Lambda_l^2/2a}-e^{-x^2\Lambda_0^2/2a}\right)^3,\end{aligned} \tag{C16}$$

where the integral over x is evaluated in $d=4$ (as this term is already of order $\tilde{g}^2_{4,l}$). For any Λ_0, Λ_l, this integral is well defined but in the limit $\Lambda_0\to\infty$, the integrand is not anymore regularized at small x and there is a logarithmic divergence. We are interested in the limit $q,\Lambda_l\ll\Lambda_0$. A simple way to isolate this divergence is to rewrite it as

$$H(q,\Lambda_0,\Lambda_l)=\frac{1}{(4\pi^2)^3}\Bigg\{-\frac{1}{2}\int_x (qx)^2 \times\frac{1}{x^6}\left(\int_a e^{-x^2\Lambda_l^2/2a}-e^{-x^2\Lambda_0^2/2a}\right)^3 + \int_x\left(e^{iqx}-1+\frac{1}{2}(qx)^2\right) \times\frac{1}{x^6}\left(\int_a e^{-x^2\Lambda_l^2/2a}-e^{-x^2\Lambda_0^2/2a}\right)^3\Bigg\}. \tag{C17}$$

The limit $\Lambda_0\to\infty$ can be taken safely in the second term, the UV divergence coming only from the first one which can be written

$$-\frac{1}{2}\int_x (qx)^2\frac{1}{x^6}\left(\int_a e^{-x^2\Lambda_l^2/2a}-e^{-x^2\Lambda_0^2/2a}\right)^3 = h\left(\frac{\Lambda_0^2}{\Lambda_l^2}\right),$$

$$h(\lambda)=-\frac{\tilde{S}_4}{8}q^2\int_0^\infty \frac{dx}{x}\left(\int_a e^{-x^2/2a}-e^{-\lambda x^2/2a}\right)^3,$$

where in the second line we performed the change of variable $x\to\Lambda_l x$ and denoted $\tilde{S}_4=2\pi^2$ the unit sphere area in dimension $d=4$. Interestingly, we have (using the variable $u=\lambda x^2$), up to terms of order λ^{-2},

$$h'(\lambda)=-\frac{3\tilde{S}_4 q^2}{16\lambda}\int_0^\infty du\int \frac{1}{2a}e^{-u/2a}\left(\int_a 1-e^{-u/2a}\right)^2$$

$$=-\frac{\tilde{S}_4 q^2}{16\lambda}\left[\left(\int_a 1-e^{-u/2a}\right)^3\right]_{u=0}^{u=\infty}+O(\lambda^{-2})=-\frac{\tilde{S}_4 q^2}{16\lambda},$$

where we have used $c(0)=\int_a=1$, which leads to $h(\lambda)\sim -(\pi^2q^2/8)\ln\lambda+O(\lambda^{-1})$. Finally, one obtains

$$H(q,\Lambda_0,\Lambda_l)=\frac{q^2}{(4\pi)^4}\left(\ln\frac{\Lambda_l}{\Lambda_0}+\chi^{(2)}(q/\Lambda_l)\right)+O\left(\frac{\Lambda_l^2}{\Lambda_0^2}\right),$$

$$\chi^{(2)}(\tilde{q})=\frac{4}{\pi^2\tilde{q}^2}\int_x\left(e^{i\tilde{q}x}-1+\frac{1}{2}(\tilde{q}x)^2\right)\frac{1}{x^6}\left(\int_a e^{-x^2/2a}\right)^3,$$

which gives [up to the factor $-\tilde{g}_{4,l}^2(N+2)/18$], the last term in Eq. (30). Using $1/x^6=1/2\int_0^\infty dt t^2 e^{-tx^2}$, one can compute the integral over x in $\chi^{(2)}(\tilde{q})$,

$$\chi^{(2)}(\tilde{q})=\frac{2}{\tilde{q}^2}\int_{a,b,c}\int_0^\infty dt\frac{t^2}{(t+\alpha_3)^2} \times\left(e^{-\tilde{q}^2/4(t+\alpha_3)}-1+\frac{\tilde{q}^2}{4(t+\alpha_3)}\right)$$

with $\alpha_3=1/2a+1/2b+1/2c$ from which we easily obtain the asymptotic behavior

$$\chi^{(2)}(\tilde{q})\sim\tilde{q}^2\int_{a,b,c}\frac{1}{48\alpha_3},\quad \tilde{q}\ll 1,$$

$$\chi^{(2)}(\tilde{q})\sim\ln\tilde{q},\quad \tilde{q}\gg 1,$$

as announced in the text (32). This yields a universal result for the η exponent. In addition, $\chi^{(2)}(\tilde{q})$ gives the scaling function of the two-point correlator $\chi^{(2)}(\tilde{q})=2\tilde{q}^2Q(\tilde{q}^2)$ where $Q(y)$ was computed in Ref. [36] in the particular case of a IR ''massive'' cutoff function of the form (7). Although our expression is more general, we have checked through series expansion that it coincides with the expression given in Ref. [36] for that choice of the cutoff.

Performing two integrations by part one can rewrite,

$$\chi^{(2)}(\tilde{q})=\int_{a,b,c>0}\frac{\hat{C}(a)\hat{C}(b)}{a^2b^2}\hat{c}(c)\frac{4}{q^2} \times\left(\frac{4}{q^2}(1-e^{-q^2/(4\alpha_3)})-\frac{1}{\alpha_3}+\frac{q^2}{8\alpha_3^2}\right) \tag{C18}$$

with $\hat{C}(a)=\int_a^\infty da'\hat{c}(a')$.

3. Quartic contribution to $\Gamma_l(\phi)$

The quartic term in Eq. (29) is obtained by inserting Eq. (18) in Eq. (13) and expanding the exponential in Eq. (13) to order zero. One gets, using Eq. (C3),

$$F_l(\phi_1,\phi_2,x)=-\int_0^l dl'\,\partial G_{l'}^x G_{l'}^x\left(\sum_i \partial_i^1\partial_i^2\right)^2 U_l(\phi_1)U_l(\phi_2)$$

$$=-\left(\frac{N+4}{(3!)^2}\phi_1^2\phi_2^2+\frac{4}{(3!)^2}(\phi_1\phi_2)^2\right) \times\int_0^l dl'\,\partial_{l'}G_{l'}^x G_{l'}^x\tilde{g}_{4l'}^2\Lambda_{l'}^{2\epsilon}, \tag{C19}$$

which is the last line in Eq. (29) [here again we have dropped terms of the form $f(\phi_i,x)$ coming from Eq. (C3)]. The integral over l' in Eq. (C19) is then treated as previously (C8). Then, when computing the Fourier transform, one obtains Eq. (35), with, using Eq. (C10),

$$\chi_l^{(4)}(q)=\int_x(e^{iqx}-1)(G_l^x)^2$$

$$=\frac{1}{16\pi^4}\int_x(e^{iqx}-1)\frac{1}{x^4}\left(\int_a e^{-x^2\Lambda_l^2/2a}-e^{-x^2\Lambda_0^2/2a}\right)^2.$$

For any Λ_l,Λ_0 finite, this function is well defined, and we see that the limit $\Lambda_0\to\infty$ is also well defined, thus

$$\chi_l^{(4)}(q)=\chi^{(4)}(q/\Lambda_l),$$

$$\chi^{(4)}(\tilde{q})=\frac{1}{16\pi^4}\int_x (e^{i\tilde{q}x}-1)\frac{1}{x^4}\left(\int_a e^{-x^2/2a}\right)^2+O\left(\frac{\Lambda_l^2}{\Lambda_0^2}\right)$$

the integral over x can be computed using $1/x^4=\int_0^\infty dtte^{-tx^2}$, one obtains

$$\chi^{(4)}(\tilde{q})=\frac{1}{16\pi^2}\int_{ab}\int_0^\infty dt\frac{t}{(t+\alpha_2)^2}(e^{-\tilde{q}^2/4(t+\alpha_2)}-1)$$

with $\alpha_2=1/2a+1/2b$, from which we extract the following asymptotic behaviors:

$$\chi^{(4)}(\tilde{q})\sim-\tilde{q}^2\int_{a,b,c}\frac{1}{128\alpha_2},\quad \tilde{q}\ll 1,$$

$$\chi^{(4)}(\tilde{q})\sim-\frac{1}{16\pi^2}\ln\tilde{q}^2,\quad \tilde{q}\gg 1,$$

as announced in the text (37), (38).

APPENDIX D: DETAILED CALCULATIONS FOR THE CO MODEL—STATICS

1. β_{g_l} function

The β function for the coupling constant g_l^K is obtained by inserting Eq. (49) in Eq. (B6). This gives straightforwardly Eq. (50) using $\partial_l G_l^{x=0}=-T/2\pi\int_0^\infty duc'(u)=T/2\pi$. One has also $G_{l'l}^{x=0}=T/2\pi(l'-l)$. Considering specifically $g_l^{1,-1}=g_l$, we first consider the possible fusion rules such that $P+Q=K_{-1,1}$:

$$P+Q=K_{1,-1}, \tag{D1}$$

$$\begin{pmatrix}\cdot\\\cdot\\1\\\cdot\\\cdot\\\cdot\\-1\\\cdot\end{pmatrix}+\begin{pmatrix}\cdot\\\cdot\\-1\\\cdot\\1\\\cdot\\\cdot\\\cdot\end{pmatrix}=\begin{pmatrix}\cdot\\\cdot\\\cdot\\\cdot\\1\\\cdot\\-1\\\cdot\end{pmatrix}, \tag{D2}$$

where $\cdot\equiv 0$, and there are $2(n-2)$ different ways to choose P,Q like that, notice $P\cdot Q=-1$. Other possible fusion rules involve charges of higher modulus, for instance, we could consider

$$P+Q=K_{1,-1}, \tag{D3}$$

$$\begin{pmatrix}\cdot\\\cdot\\1\\\cdot\\-2\\\cdot\\1\\\cdot\end{pmatrix}+\begin{pmatrix}-1\\\cdot\\-1\\\cdot\\2\\\cdot\\\cdot\\\cdot\end{pmatrix}=\begin{pmatrix}-1\\\cdot\\\cdot\\\cdot\\\cdot\\\cdot\\1\\\cdot\end{pmatrix} \tag{D4}$$

with $P^2=Q^2=6$.

It is then useful to write the integrals $\tilde{J}_l^{(1)}$ and $\tilde{J}_{l,l'}^{(2)}$ in Eq. (50) in terms of the variables $\tilde{x}=\Lambda_l x$ and $\mu=l-l'$. Using Eq. (56), and specifying to g_l, one has

$$\frac{\tilde{J}_l^{(1)}}{2T^2}\sum_{P,Q}{}' g_l^P g_l^Q (P\cdot Q)^2=(n-2)g_l^2\int_{\tilde{x}}\partial\gamma_0(\tilde{x})\gamma_l(\tilde{x}) \tag{D5}$$

and

$$\frac{-1}{2T^2}\sum_{P,Q}{}'(P.Q)^3\int_0^l dl'\tilde{J}_{l,l'}^2 g_{l'}^P g_{l'}^Q=(n-2)T\int_{\tilde{x}}[\partial\gamma_0(\tilde{x})-\partial\gamma_0(0)]\int_0^l d\mu\,\partial\gamma_\mu(\tilde{x})[\gamma_l(\tilde{x})-\gamma_\mu(\tilde{x})]e^{(4-T/\pi)\mu}e^{T\gamma_\mu(\tilde{x})}g_{l-\mu}^2 \tag{D6}$$

with $\Sigma'_{P,Q}\equiv\Sigma_{P,Q,P+Q=K}$. We study the flow near $T_c=4\pi$, and as Eq. (D6) is already of order $g_{l-\mu}^2$, we can evaluate the integral over μ exactly at T_c: in particular, $e^{(4-T/\pi)\mu}=1+O(\tau)$. Moreover, as the integral is convergent, it is dominated by the vicinity of the fixed point $\mu=0$. We can then substitute in Eq. (D6) $g_{l-\mu}$ by g_l. The remaining integral over μ is then straightforwardly computed by integration by parts. Equation (D5) together with Eq. (D6), integrated over μ and using Eq. (D9), then lead in the limit $n\to 0$ to Eq. (55),

$$\partial_l g_l = \left(2-\frac{T}{2\pi}\right) g_l - 2 g_l^2 \partial\gamma_0(0) \int_{\tilde{x}} \gamma_l(\tilde{x}) - \frac{2g_l^2}{T_c}\int_{\tilde{x}} [\partial\gamma_0(\tilde{x}) - \partial\gamma_0(0)](e^{T_c\gamma_l(\tilde{x})}-1). \quad \text{(D7)}$$

To compute the integrals over $\tilde{x}$ in Eq. (D7) in the limit $l\to\infty$ at $T=T_c$ we first quote some useful relations. Using the decomposition of the cutoff function (17), we have

$$\partial\gamma_\mu(x) = \frac{1}{2\pi}\int_a e^{-x^2 e^{2\mu}/(2a)},$$

$$\gamma_\mu(x) = \frac{1}{4\pi}\int_a \int_{x^2/2a}^{x^2e^{2\mu}/(2a)} \frac{dy}{y} e^{-y} \quad \text{(D8)}$$

and the following identities:

$$\partial_\mu \gamma_\mu(x) = \partial\gamma_\mu(x),$$

$$2x^2\partial_{x^2}\gamma_\mu(x) = \partial\gamma_\mu(x) - \partial\gamma_{\mu=0}(x). \quad \text{(D9)}$$

We first compute these integrals in the semibounded domain $|x|>\epsilon$ and then take the limit $\epsilon\to 0$, in order to avoid problems of convergence (the limit $l\to\infty$ does not introduce any problem). Let us decompose the integrals over $\tilde{x}$ in the following way, writing $\mathcal{B}_\infty/2$ as

$$\partial\gamma_0(0)\int_x' \gamma_l(x) + \frac{1}{T_c}\int_x' (\partial\gamma_0(x)-\partial\gamma_0(0))(e^{T_c\gamma_\infty(x)}-1)$$
$$= \frac{1}{T_c}\int_x' \partial\gamma_0(x) e^{T_c\gamma_\infty(x)} - \frac{1}{T_c}\int_x' \partial\gamma_0(0)(e^{T_c\gamma_\infty(x)}-1) + \int_x' \partial\gamma_0(0)\gamma_\infty(x) - \frac{\partial\gamma_0(x)}{T_c} \quad \text{(D10)}$$

with $\int_x' \equiv \int_{|x|>\epsilon}$. Using the previous formula (D9) for $l\to\infty$, $2x^2\partial_{x^2}\gamma_\infty(x) = -\partial\gamma_{\mu=0}(x), x>0$ since $\partial\gamma_\infty(x)=0, x>0$, together with $\partial\gamma_0(0)=2/T_c$ we are left with (performing the change of variable $u=x^2$), and denoting $\gamma_\infty(x) = \tilde{\gamma}_\infty(x^2)$,

$$\partial\gamma_0(0)\int_x' \gamma_l(x) + \frac{1}{T_c}\int_x' (\partial\gamma_0(x)-\partial\gamma_0(0))(e^{T_c\gamma_\infty(x)}-1)$$
$$= \frac{-2\pi}{T_c^2}\int_\epsilon^\infty du[uT_c\partial_u\tilde{\gamma}_\infty(u)e^{T_c\tilde{\gamma}_\infty(u)} + (e^{T_c\tilde{\gamma}_\infty(u)}-1)] + \frac{2\pi}{T_c}\int_\epsilon^\infty du[\tilde{\gamma}_\infty(u) + u\partial_u\tilde{\gamma}_\infty(u)]$$
$$= \frac{-2\pi}{T_c^2}\{u[e^{T_c\tilde{\gamma}_\infty(u)} - 1 - T_c\tilde{\gamma}_\infty(u)]\}_\epsilon^\infty, \quad \text{(D11)}$$

as one recognizes total derivatives in the integrands. Using explicitly Eq. (D8),

$$\tilde{\gamma}_\infty(u) = \frac{1}{4\pi}\int_a E_1(u/(2a)), \quad \text{(D12)}$$

where $E_1(z) = -\mathrm{Ei}(-z) = \int_z^{+\infty} e^{-z}/z$, with $\mathrm{Ei}(x)$ the exponential integral function, behaves asymptotically as

$$E_1(z) \sim -\gamma_E - \ln z + O(z), \quad z\ll 1, \quad \text{(D13)}$$

$$E_1(z) \sim \frac{e^{-z}}{z}[1+O(1/z)], \quad z\gg 1, \quad \text{(D14)}$$

where γ_E is the Euler constant, the limit $\epsilon\to 0$ in Eq. (D11) can be taken safely to obtain

$$\mathcal{B}_\infty = \frac{4\pi}{T_c^2}\exp\left[-\left(\gamma_E - \int_a \ln 2a\right)\right], \quad \text{(D15)}$$

which leads, together with Eq. (55), to the fixed point value g^* (57).

2. Bilocal term for CO model

We compute in this section the bilocal part in the effective action given by Eq. (59). Performing in Eq. (59) the change of variable $l'\to\mu=l-l'$ and using the notations (56) and $G_{l'}^x = -T(\gamma_\mu(\tilde{x}) - \gamma_l(\tilde{x}))$, one gets

$$\hat{V}_l^{K,Pq} = \frac{1}{2}\int_x (e^{iqx}-1)\hat{F}_l^{K,Px},$$

$$\hat{F}_l^{K,Px} = \frac{\Lambda_l^4}{T^2}(KP)^2\int_0^l d\mu\, \partial\gamma_\mu(\Lambda_l x)[\gamma_\mu(\Lambda_l x) - \gamma_l(\Lambda_l x)] e^{-TK.P\gamma_\mu(\Lambda_l x)} e^{(4-T/\pi)\mu} g_{l-\mu}^2. \quad \text{(D16)}$$

As previously, this integral is already of order $g_{l-\mu}^2$, so it can be evaluated at T_c, in particular, $e^{(4-T/\pi)\mu} = 1 + O(\epsilon)$. Besides, the integral over μ is convergent and dominated by $\mu=0$, so that we substitute $g_{l-\mu}^2$ by g_l^2. The remaining integral over μ is then straightforwardly computed to obtain

$$\hat{F}_l^{K,Px} = -\frac{\Lambda_l^4}{T^2}g_l^2\left(\frac{1}{T^2}(e^{-T_cK\cdot P\gamma_l(\Lambda_l x)}-1) + K\cdot P\frac{\gamma_l(\Lambda_l x)}{T}\right), \quad \text{(D17)}$$

where Eq. (D8) can be written as

$$\gamma_l(\Lambda_l x) = \frac{1}{4\pi}\int_a E_1(x^2\Lambda_l^2/2a) - E_1(x^2\Lambda_0^2/2a) \quad \text{(D18)}$$

with the asymptotic behaviors of $E_1(z)$ given in Eqs. (D13) and (D14). For any Λ_l, Λ_0 the integral over x in Eq. (D16) is well defined, but we see that in the limit $\Lambda_0\to\infty$ (i.e., $\Lambda_l, q\ll\Lambda_0$), there is a logarithmic divergence (for small x) and only for charges such that $K\cdot P=-2$. Indeed, at small x, using Eq. (D13), $-T_cK\cdot P\gamma_l(\Lambda_l x)\sim K\cdot P[\gamma_E + \ln(\Lambda_l^2x^2)]$, leading to $e^{-T_cK\cdot P\gamma_l(\Lambda_l x)}\sim x^{2K\cdot P}$. This implies that the limit

$\Lambda_0 \to \infty$ only diverges for $K\cdot P = -2$ (there is no problem with the large x behavior as the integrand decay exponentially for any couple of charges we consider here).

a. The case of charges K·P=1 or 2

For these charges, the limit $\Lambda_0 \to \infty$ can be taken directly on Eq. (D16). This leads to, performing the change of variable $x \to \Lambda_l x$ and the integral over the angular variable on Eq. (D16),

$$\hat{V}_l^{K,Pq} = -q^2 g_l^2 \frac{\pi}{T_c^4} \int_0^\infty dr \frac{r}{\tilde{q}^2} [J_0(|\tilde{q}|r) - 1] \times \left[\exp\left(-K\cdot P \int_a E_1(r^2/2a) \right) - 1 + K\cdot P \int_a E_1(r^2/2a) \right], \tag{D19}$$

where $\tilde{q} = q/\Lambda_l$ and $J_0(z)$ is a Bessel function of the first kind. This defines the function $\chi^{K,P}(k)$ in that case,

$$\chi^{K,P}(k) = 4 \exp\left[2\left(\gamma_E - \int_a \ln 2a \right) \right] \int_0^\infty dr \frac{r}{k^2} [J_0(|k|r) - 1] \times \left[\exp\left(-K\cdot P \int_a E_1(r^2/2a) \right) - 1 + K\cdot P \int_a E_1(r^2/2a) \right]. \tag{D20}$$

The small k behavior [the first line of Eq. (63)] is straightforwardly obtained as

$$\chi^{K,P}(k) \sim a_{K\cdot P} + O(k^2),$$

$$a_{K\cdot P} = -\frac{\exp\left[2\left(\gamma_E - \int_a \ln 2a \right) \right]}{2} \times \int_{u>0} u \left[\exp\left(-K\cdot P \int_a E_1(u/2a) \right) - 1 + K\cdot P \int_a E_1(u/2a) \right], \tag{D21}$$

where we performed the change of variable $u = r^2$.

For $K\cdot P = 1$ or 2, $r \exp[-K\cdot P \int_a E_1(r^2/2a)] \sim r^{2K\cdot P+1}$ when $r \ll 1$ is analytic in 0 and using $J_0(k) \sim k^{-1/2} \cos(k - \pi/4)$ one finds for $k \gg 1$,

$$\int_0^\infty dr \frac{r}{k^2} J_0(|k|r) \left[\exp\left(-K\cdot P \int_a E_1(r^2/2a) \right) - 1 \right] \sim O\left(\frac{1}{k^{5/2}} \right). \tag{D22}$$

We have, moreover,

$$\int_0^\infty dr \frac{r}{k^2} [J_0(|k|r) - 1] \int_a E_1(r^2/2a) = \int_a \frac{2 - 2e^{-ak^2/2} - ak^2}{k^4}. \tag{D23}$$

Using Eq. (D23) together with Eq. (D22), one obtains the leading behavior of $\chi^{K,P}(k)$ (D20) in the large k limit, i.e., the first line of Eq. (64),

$$\chi^{K,P}(k) \sim b_{K\cdot P} \frac{1}{k^2},$$

$$b_{K\cdot P} = -2 \exp\left[2\left(\gamma_E - \int_a \ln 2a \right) \right] \times \left(\int_{u>0} \left\{ \exp\left[-K\cdot P \int_a E_1\left(\frac{u}{2a} \right) \right] - 1 \right\} - 2K\cdot P c'(0) \right), \tag{D24}$$

where we made the change of variable $u = r^2$ and used $c'(0) = -\int_a a$.

b. The case of charges K·P=−1

In that case, $\chi^{K,P}(k)$ is formally obtained as previously (D20), the small k behavior being still given by Eq. (D21). However, the large k behavior is dominated by the small r region and as noticed previously for $r \ll 1$, $r[\exp(-K\cdot P \int_a E_1(r^2/2a)) - 1] \sim r^{2K\cdot P+1} = r^{-1}$, which leads to a logarithmic divergence in the large k limit. It can be obtained by computing

$$\int_0^\infty dr\, r [J_0(kr) - 1] \left[\exp\left(\int_a E_1(r^2/2a) \right) - 1 \right] \sim \exp\left(-\gamma_E + \int_a \ln 2a \right) \int_0^\infty \frac{dr}{r} [J_0(kr) - 1] \sim -\exp\left(-\gamma_E + \int_a \ln 2a \right) \ln k, \quad k \gg 1. \tag{D25}$$

The last term in Eq. (D20) has the same behavior (D23) independently of $K\cdot P$ and Eq. (D25), together with Eq. (D20), for $K\cdot P = -1$ leads to the second line of Eq. (64),

$$\chi^{K,P}(k) \sim b_{-1} \frac{\ln k}{k^2},$$

$$b_{-1} = -4 \exp\left(\gamma_E - \int_a \ln 2a \right). \tag{D26}$$

c. The case of charges $K\cdot P=-2$

As pointed out previously, there is in that case a logarithmic divergence when $\Lambda_0 \gg 1$. We isolate this divergence by writing

$$\hat{V}_l^{K\cdot Pq} = \frac{-1}{4}\int_x (qx)^2 \hat{F}_l^{K\cdot Px} + \frac{1}{2}\int_x \left(e^{iqx} - 1 + \frac{1}{2}(qx)^2 \right) \hat{F}_l^{K\cdot Px}, \tag{D27}$$

the second term being well defined in the limit $\Lambda_0 \to \infty$. We focus now on the first part, using the explicit expression of $\gamma_l(\Lambda_l x)$ (D8),

$$\begin{aligned} -\frac{1}{4}\int_x (qx)^2 \hat{F}_l^{K-Kx} &= \frac{1}{8T_c^4} g_l^2 q^2 \int_x x^2 \left\{ \exp\left[2\int_a E_1\left(\frac{x^2}{2a}\right) - E_1\left(\frac{x^2\Lambda_0^2}{2a\Lambda_l^2}\right)\right] \right. \\ &\quad \left. -1 - 2\left[\int_a E_1\left(\frac{x^2}{2a}\right) - E_1\left(\frac{x^2\Lambda_0^2}{2a\Lambda_l^2}\right)\right]\right\} = \mathcal{H}\left(\frac{\Lambda_0^2}{\Lambda_l^2}\right), \end{aligned} \tag{D28}$$

where we made the change of variable $x \to \Lambda_l x$. To analyze the large argument behavior of $\mathcal{H}(\lambda)$, we take the derivative with respect to λ

$$\begin{aligned} \mathcal{H}'(\lambda) &= \frac{\pi}{4T_c^4} g_l^2 q^2 \int_0^\infty dx x^3 \int_a \frac{2}{\lambda} e^{-\lambda x^2/2a} \\ &\quad \times \left(\exp\left\{ 2\left[\int_a E_1\left(\frac{x^2}{2a}\right) - E_1\left(\frac{\lambda x^2}{2a}\right)\right]\right\} - 1\right), \end{aligned} \tag{D29}$$

where we have used $E_1'(z) = -e^{-z}/z$. Making the change of variable $u=\lambda x^2$ in $\mathcal{H}'(\lambda)$ one obtains

$$\begin{aligned} \mathcal{H}'(\lambda) &= \frac{\pi}{8T_c^4} g_l^2 q^2 \int_0^\infty \frac{du}{\lambda^3} u \int_a 2e^{-u/2a}\left(\exp\left\{ 2\left[\int_a E_1\left(\frac{u}{2a\lambda}\right) \right.\right.\right. \\ &\quad \left.\left.\left. - E_1\left(\frac{u}{2a}\right)\right]\right\} - 1\right), \end{aligned}$$

using the large λ behavior $E_1(u/(2a\lambda)) \sim -\gamma_E + \int_a \ln(2a) - \ln(u/\lambda) + O(1/\lambda)$ one gets

$$\begin{aligned} \mathcal{H}'(\lambda) &= \frac{\pi}{8T_c^4} g_l^2 q^2 \exp\left(2\int_a (-\gamma_E + \ln 2a)\right)\frac{1}{\lambda}, \\ &\int_0^\infty du \int_a \frac{2}{u} e^{-u/2a} \exp\left[-2\int_a E_1\left(\frac{u}{2a}\right)\right] + O(\lambda^{-2}) \\ &= \frac{\pi}{8T_c^4} g_l^2 q^2 \exp\left(2\int_a (-\gamma_E + \ln 2a)\right) \\ &\quad \times \frac{1}{\lambda}\left\{ \exp\left[-2\int_a E_1\left(\frac{u}{2a}\right)\right]\right\}_0^\infty + O(\lambda^{-2}) \\ &= \frac{\pi}{8T_c^4} g_l^2 q^2 \exp\left(2\int_a (-\gamma_E + \ln 2a)\right)\frac{1}{\lambda} + O(\lambda^{-2}), \end{aligned} \tag{D30}$$

where we have used the asymptotic behaviors (D13), (D14). This leads finally to

$$\begin{aligned} -\frac{1}{4}\int_x (qx)^2 \hat{F}_l^{K,Px} &= -\delta_{K,-P} A_l q^2 \ln\left(\frac{\Lambda_l}{\Lambda_0}\right), \\ +O(\Lambda_l^2/\Lambda_0^2) A_l &= \frac{\pi}{4T_c^4} g_l^2 \exp\left(2\int_a (-\gamma_E + \ln 2a)\right), \end{aligned} \tag{D31}$$

which is the first term in Eq. (61) with the amplitude A_l given in Eq. (62).

In the second line of Eq. (D27), we perform the change of variable $x \to \Lambda_l x$ and the integral over the angular variable to get

$$\begin{aligned} \frac{1}{2}\int_x \left(e^{iqx} - 1 + \frac{1}{2}(qx)^2\right) \hat{F}_l^{K,Px} &= -q^2 g_l^2 \frac{\pi}{T_c^4} \int_0^\infty dr \frac{r}{\tilde{q}^2}\left(J_0(|\tilde{q}|r) - 1 + \frac{1}{4}\tilde{q}^2 r^2\right) \\ &\quad \times \left[\exp\left(2\int_a E_1(r^2/2a)\right) - 1 - 2\int_a E_1(r^2/2a)\right], \end{aligned} \tag{D32}$$

where $J_0(z)$ is a Bessel function of the first kind, from which we get the function $\chi^{K,P}(k)$ defined in the text for $K\cdot P=-2$,

$$\begin{aligned} \chi^{K,P}(k) &= 4\exp\left[2\left(\gamma_E - \int_a \ln 2a\right)\right] \int_0^\infty dr \frac{r}{k^2} \\ &\quad \times \left(J_0(|k|r) - 1 + \frac{1}{4}k^2 r^2\right)\left[\exp\left(2\int_a E_1(r^2/2a)\right) \right. \\ &\quad \left. - 1 - 2\int_a E_1(r^2/2a)\right]. \end{aligned} \tag{D33}$$

The small k behavior [i.e., the second line of Eq. (63) in the text] is easily obtained

$$\chi^{K,P}(k)\sim a_{-2}k^2, \quad k\ll 1,$$

$$a_{-2}=\frac{\exp\left[2\left(\gamma_E-\int_a \ln 2a\right)\right]}{32}\int_0^\infty du u^2 \times\left[\exp\left(2\int_a E_1(u/2a)\right)-1-2\int_a E_1(u/2a)\right], \tag{D34}$$

where we made the change of variable $u=r^2$. The large k behavior is governed by the small r region in the integral (D33), where $r(\exp[2\int_a E_1(r^2/2a)]-1-2\int_a E_1(r^2/2a))\sim\exp(-2\gamma_E+\int_a \ln 2a)r^{-3}$, which implies for $k\gg 1$,

$$\chi^{K,P}(k)\sim\frac{4}{k^2}\int_0^\infty\frac{dr}{r^3}\left(J_0(|k|r)-1+\frac{1}{4}k^2r^2\right)+O(1) \sim\ln k+O(1), \tag{D35}$$

which is the last line of Eq. (64) in the text.

APPENDIX E: DETAILED CALCULATIONS FOR THE CO MODEL—EQUILIBRIUM DYNAMICS

1. Derivation of the RG flow

We restrict our analysis to order 1 $O(U_l)$, and at this order the RG flow reads (14)

$$\partial_l U_l(u,i\hat{u})=\frac{1}{2}\partial G^{x=0}_{l,ij}\partial_i\partial_j U_l(u,i\hat{u}), \tag{E1}$$

where $U_l(u,i\hat{u})$ is given by Eq. (80) and the indices i,j formally refer to the components of the vector ϕ (73) and the time dependence, i.e., $\partial_i\equiv\delta/\delta u_t,\delta/\delta i\hat{u}_t$. From Eq. (74), the matrix $G^q_{l,tt'}$ has the following expression:

$$G^q_l=\begin{pmatrix} C^q_l & R^q_l \\ R^{q\dagger}_l & 0\end{pmatrix}. \tag{E2}$$

With these notations, we have

$$\frac{1}{2}\partial G^{x=0}_{l,ij}\partial_i\partial_j=\frac{1}{2}\frac{\delta}{\delta u}\partial C^{x=0}_l\frac{\delta}{\delta u}+\frac{\delta}{\delta u}\partial R^{x=0}_l\frac{\delta}{\delta i\hat{u}}, \tag{E3}$$

where we will often use the matrix notation for time, i.e., $uv=\int_t u_t v_t$. Acting with this operator on $U_l(u,i\hat{u})$, one gets

$$\begin{aligned}\frac{1}{2}\partial G^{x=0}_{l,ij}\partial_i\partial_j&\left(\int_t i\hat{u}_{xt}F_{lt}(u)-\frac{1}{2}\int_{tt'}i\hat{u}_{xt}i\hat{u}_{xt'}\Delta_{ltt'}(u)\right)\\ &=-\frac{1}{4}\int_{tt'}i\hat{u}_t i\hat{u}_{t'}\int_{t_1t_1'}\frac{\delta}{\delta u_{t_1}}\partial C^{x=0}_{lt_1t_1'}\frac{\delta}{\delta u_{t_1'}}\Delta_{ltt'}(u)\\ &\quad+\frac{1}{2}\int_t i\hat{u}_t\int_{t_1t_1'}\frac{\delta}{\delta u_{t_1}}\partial C^{x=0}_{lt_1t_1'}\frac{\delta}{\delta u_{t_1'}}F_{lt}(u)\\ &\quad-\int_t i\hat{u}_t\int_{t_1>t_1'}\partial R^{x=0}_{lt_1t_1'}\frac{\delta}{\delta u_{t_1}}\Delta_{ltt_1'}\\ &\quad+\int_{t_1>t}\partial R^{x=0}_{lt_1t}\frac{\delta}{\delta u_{t_1}}F_t(u).\end{aligned} \tag{E4}$$

The last term vanishes by causality since $F_t(u)$ depends on u_{t_1} with $t_1<t$ only. Identifying in Eq. (E1) the coefficient of the powers in the field $i\hat{u}$ one gets Eq. (84) in the text.

The first equation of Eq. (84) is easily solved, and it gives

$$\begin{aligned}\Delta_{ltt'}(u)&=\exp\left(\frac{1}{2}\int_{t_1t_1'}\frac{\delta}{\delta u_{t_1}}C^{x=0}_{lt_1t_1'}\frac{\delta}{\delta u_{t_1'}}\right)\Delta_{l=0tt'}(u)\\ &=2e^{-C^{x=0}_{l0}+C^{x=0}_{lt-t'}}\Lambda_0^2 g_0\cos(u_{xt}-u_{xt'}),\end{aligned} \tag{E5}$$

where we have used $C^{x=0}_{l=00}=0$. From the previous study of the statics, one has that

$$e^{-C^{x=0}_{l0}}\Lambda_0^2 g_0=\Lambda_l^2 g_l+O(g_l^2), \tag{E6}$$

which leads together with Eq. (E5) to the first line in Eq. (86). By taking the functional derivative with respect to $u_{xt'}$ in the second line of Eq. (84) and using the same manipulations one gets the second line in Eq. (86).

2. Computation of the dynamical exponent z

Here we compute the self-energy $\Sigma_{l\omega}$ (89) given by

$$\Sigma_{l\omega}=\int_0^\infty dt e^{i\omega t}\Sigma_{lt},$$

$$\Sigma_{lt}=-2\Lambda_l^2 g_l\left(R^{x=0}_{lt}e^{C^{x=0}_{lt}}-\delta(t)\int_0^\infty dt' R^{x=0}_{lt'}e^{C^{x=0}_{lt'}}\right). \tag{E7}$$

Notice the terms proportional to $\delta(t)$ in Σ_{lt} [not given in the text (86) for clarity] which guarantees that $\Sigma_{l\omega=0}=0$, and with the explicit expressions for the bare correlation and response functions (75) computed with the cutoff decomposition (17),

$$C_{lt}^{x=0}=\frac{T}{4\pi}\int_a \ln\left(\frac{|t|+\frac{a}{2\Lambda_l^2}}{|t|+\frac{a}{2\Lambda_0^2}}\right), \tag{E8}$$

$$R_{lt}^{x=0}=\frac{\theta(t)}{4\pi}\int_a \frac{1}{t+\frac{a}{2\Lambda_0^2}}-\frac{1}{t+\frac{a}{2\Lambda_l^2}}. \tag{E9}$$

After an integration by part in Eq. (E7) and using those explicit expressions one gets to order τ,

$$\Sigma_{l\omega}=-\frac{2\Lambda_l^2 g_l}{T_c}i\omega\int_0^\infty dt e^{i\omega t}\times\left\{\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2\Lambda_l^2}}{t+\frac{a}{2\Lambda_0^2}}\right)\right]-1\right\}. \tag{E10}$$

This expression is logarithmically divergent for $\Lambda_0\to\infty$ (the integrand behaves as $1/t$ at small t in this limit), and a way to isolate this divergence is to decompose this integral in the following way (and performing the change of variable $t\to t/\Lambda_l^2$):

$$\begin{aligned}\Sigma_{l\omega}=&-\frac{2g_l}{T_c}i\omega\int_1^\infty dt e^{i\tilde{\omega}t}\left\{\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]-1\right\}\\&-\frac{2g_l}{T_c}i\omega\int_0^1 dt(e^{i\tilde{\omega}t}-1)\\&\times\left\{\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]-1\right\}\\&-\frac{2g_l}{T_c}i\omega\int_0^1 dt\left\{\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]-1\right\},\end{aligned} \tag{E11}$$

where $\tilde{\omega}=\omega/\Lambda_l^2$ and $\lambda=\Lambda_l^2/\Lambda_0^2$. In the first two lines we can take safely the limit $\lambda\to 0$ and we focus now on the divergent part of the last term,

$$\mathbb{H}(\lambda)=-\frac{2g_l}{T_c}i\omega\int_0^1 dt\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]. \tag{E12}$$

Taking the derivative with respect to λ, one has

$$\begin{aligned}\mathbb{H}'(\lambda)=&-\frac{2g_l}{T_c}i\omega\int_0^1 dt\int_a\frac{-a}{2}\frac{1}{t+\frac{\lambda a}{2}}\\&\times\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]\\=&-\frac{\mathbb{H}(\lambda)}{\lambda}-\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\int_0^1 dtt\int_a\frac{1}{t+\frac{a}{2}}\\&\times\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]\\&-\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\int_0^1 dtt\int_a\left(\frac{1}{t+\frac{\lambda a}{2}}-\frac{1}{t+\frac{a}{2}}\right)\\&\times\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right].\end{aligned} \tag{E13}$$

In the integral of the second line, we can take the limit $\lambda\to 0$, it gives

$$\begin{aligned}&-\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\int_0^1 dtt\int_a\frac{1}{t+\frac{a}{2}}\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]\\&\sim-\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\int_0^1\int_a\frac{1}{t+\frac{a}{2}}\exp\left[\int_a \ln t+\frac{a}{2}\right]+O(1)\\&\sim-\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\left\{\exp\left[\int_a \ln\left(1+\frac{a}{2}\right)\right]-\exp\left[\int_a \ln\left(\frac{a}{2}\right)\right]\right\}\\&+O(1).\end{aligned}$$

The last term in Eq. (E13) can be integrated by parts, to get

$$-\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\int_0^1 dt t\int_a \frac{1}{t+\frac{\lambda a}{2}}$$

$$-\frac{1}{t+\frac{a}{2}}\exp\left[\int_a \ln\left(\frac{t+\frac{a}{2}}{t+\frac{\lambda a}{2}}\right)\right]$$

$$=\frac{\mathbb{H}(\lambda)}{\lambda}+\frac{2g_l}{T_c}\frac{i\omega}{\lambda}\exp\left[\int_a \ln\left(1+\frac{a}{2}\right)\right]+O(1). \tag{E14}$$

Finally, $\mathbb{H}'(\lambda)$ in Eq. (E13) can be written as

$$\mathbb{H}'(\lambda)\sim i\omega\left[\frac{g_l}{2T_c}\exp\left(\int_a \ln 2a\right)\frac{1}{\lambda}+O(1)\right],$$

$$\mathbb{H}(\lambda)\sim i\omega\left[\frac{g_l}{2T_c}\exp\left(\int_a \ln 2a\right)\ln\lambda+O(1)\right], \tag{E15}$$

which gives together with the last line of Eq. (E11) the first term in Eq. (91) with the amplitude $B_l = g_l/2T_c \exp(\int_a \ln 2a)$. The first two lines of Eq. (E11), where we take the limit $\lambda \to 0$ define the function $\chi^{(\mathrm{dyn})}(\nu)$ of Eq. (91),

$$\chi^{(\mathrm{dyn})}(\nu)=-4\exp\left(-\int_a \ln 2a\right)\left(\int_1^\infty dt e^{i\nu t}\right.$$

$$\times\left\{\frac{1}{t}\exp\left[\int_a \ln\left(t+\frac{a}{2}\right)\right]-1\right\}+\int_0^1 dt(e^{i\nu t}-1)$$

$$\left.\times\left\{\frac{1}{t}\exp\left[\int_a \ln\left(t+\frac{a}{2}\right)\right]-1\right\}\right). \tag{E16}$$

The small argument behavior of $\chi^{(\mathrm{dyn})}(\nu)$ is dominated by the large t region of the integrand [i.e., the first line of Eq. (E16)]. Using that $(1/t\exp[\int_a \ln(t+a/2)]-1)\sim\int_a a/2t$ for $t \gg 1$, one gets

$$\chi^{(\mathrm{dyn})}(\nu)\sim -4\exp\left(-\int_a \ln 2a\right)\int_a \frac{a}{2}\int_1^\infty e^{i\nu t}\frac{1}{t},$$

$$\nu\ll 1\sim 4\exp\left(-\int_a \ln 2a\right)\int_a \frac{a}{2}\ln\nu, \quad \nu\ll 1, \tag{E17}$$

which is the asymptotic behavior announced in the text (93) with the nonuniversal amplitude $a_{\mathrm{dyn}} = 4\exp(-\int_a \ln 2a)\int_a a/2$. The large ν behavior of $\chi^{(\mathrm{dyn})}(\nu)$ is governed by the small t region of the integrand, i.e., the second line of Eq. (E16):

$$\chi^{(\mathrm{dyn})}(\nu)\sim -4\exp\left(-\int_a \ln 2a\right)\exp\left(\int_a \ln\frac{a}{2}\right)$$

$$\times\int_0^1 dt(e^{i\nu t}-1)\frac{1}{t}$$

$$\sim \ln\nu \quad \nu\gg 1, \tag{E18}$$

which is the asymptotic behavior announced in the text (94).

We show here how to take directly, in a cruder way, the limit $l\to\infty$ in Σ_{lt} (E7). Indeed, using the explicit expression of $C_{lt}^{x=0}$ and $R_{lt}^{x=0}$ (E8), one has

$$C_{lt}^{x=0}\sim -\frac{T}{4\pi}\int_a [\ln(4\Lambda_0^2 t+2a)-\ln(2a)-2l]+O(e^{-2l}),$$

$$R_{lt}^{x=0}\sim\frac{1}{4\pi}\int_a \frac{1}{t+\frac{a}{2\Lambda_0^2}}+O(e^{-2l}). \tag{E19}$$

This allows to take the large l limit in Σ_{lt} at T_c (as it is already of order τ)

$$\lim_{l\to\infty}\Sigma_{lt}=-\frac{\Lambda_0^2}{2\pi}g^*\exp\left(\int_a \ln(2a)\right)\int_a \frac{1}{t+\frac{a}{2\Lambda_0^2}}$$

$$\times\exp\left(-\int_a \ln(4\Lambda_0^2 t+2a)\right) \tag{E20}$$

for $t>0$. We then obtain directly $\Sigma_{l\omega}$ in the limit $l\to\infty$ as

$$\lim_{l\to\infty}\Sigma_{l\omega}=-\frac{g^*\Lambda_0^2\exp\left(\int_a \ln(2a)\right)}{2\pi}i\omega\int_0^\infty dt e^{i\omega t}$$

$$\times\exp\left(-\int_a \ln 4\Lambda_0^2 t+2a\right)$$

$$=-\frac{g^*\exp\left(\int_a \ln(2a)\right)}{2\pi}i\omega\int_0^\infty dt e^{i(\omega/\Lambda_0^2 t)}$$

$$\times\exp\left(-\int_a \ln 4t+2a\right). \tag{E21}$$

The small ω/Λ_0^2 behavior is governed by the large t region of the integrand, which gives

$$\lim_{l\to\infty}\Sigma_{l\omega}\sim -\frac{g^* \exp\left(\int_a \ln(2a)\right)}{2\pi} i\omega\int_1^\infty \exp i(\omega/\Lambda_0^2)t\frac{1}{4t}$$

$$\sim B^* i\omega \ln\left(\frac{\omega}{\Lambda_0^2}\right)+O\left(\frac{i\omega}{\Lambda_0^2}\right),\quad \frac{i\omega}{\Lambda_0^2}\ll 1, \tag{E22}$$

which gives the same result obtained by the previous analysis (95).

3. Scaling function at equilibrium

In this section, we show how to solve the equation for the response function (102). First, it is natural to search for a solution under the form $\mathcal{R}_t^q = e^{-q^2 t}\mathcal{G}_t^q$. Then, performing the change of variable $u=t-t_1$ and using the explicit expression (E20), one gets the following equation for $\mathcal{G}_t^q$:

$$\partial_{\tilde{t}}\mathcal{G}_{\tilde{t}}^{\tilde{q}} = 4B^*\int_0^{\tilde{t}} du\int_a \frac{1}{u+\frac{a}{2}}\exp\left(-\int_a \ln(4u+2a)\right)e^{\tilde{q}^2 u}$$

$$-4B^*\int_0^\infty du\int_a \frac{1}{u+\frac{a}{2}}\exp\left(-\int_a \ln(4u+2a)\right), \tag{E23}$$

where $\tilde{q}=q/\Lambda_0$ and $\tilde{t}=\Lambda_0^2 t$, with the initial conditions:

$$\mathcal{G}_{0^+}^{\tilde{q}}=1, \tag{E24}$$

$$\mathcal{G}_0^{\tilde{q}}=0. \tag{E25}$$

The second term in the left-hand side is a total derivative and can be integrated. Performing an integration by part on the first term, one gets

$$\mathcal{G}_{\tilde{t}}^{\tilde{q}} = 1+4B^*\left[\tilde{q}^2\int_0^{\tilde{t}} dv\int_0^v du \exp\left(-\int_a \ln(4u+2a)\right)e^{q^2 u} - \int_0^{\tilde{t}} dv\, e^{q^2 v}\exp\left(-\int_a \ln(4v+2a)\right)\right]. \tag{E26}$$

Performing an integration by part in the integral over v on the first integral and performing the change of variable $u' = q^2 u$ in the remaining integrals, one gets

$$\mathcal{G}_{\tilde{t}}^{\tilde{q}} = 1+4B^*\left\{(\tilde{q}^2\tilde{t}-1)\frac{1}{\tilde{q}^2}\int_0^{\tilde{q}^2\tilde{t}} du \exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right]\right.$$

$$\times(e^u-1)-\frac{1}{\tilde{q}^2}\int_0^{\tilde{q}^2\tilde{t}} du\, u e^u \exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right] \tag{E27}$$

$$\left.+(\tilde{q}^2\tilde{t}-1)\frac{1}{\tilde{q}^2}\int_0^{\tilde{q}^2\tilde{t}} du \exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right]\right\}. \tag{E28}$$

We now want to find the scaling function, i.e., the asymptotic behavior when $\tilde{q}\to 0$ ($\Lambda_0\to\infty$), keeping $\tilde{q}^2\tilde{t}=y$ fixed. In the two first lines of the above expression, the limit $\tilde{q}\to 0$ can be taken safely, although the last term is divergent in this limit. Thus, one has

$$\mathcal{G}_{\tilde{t}}^{\tilde{q}} = 1+4B^*\left\{(\tilde{q}^2\tilde{t}-1)\int_0^{\tilde{q}^2\tilde{t}} du\frac{e^u-1}{4u}-\frac{1}{4}\int_0^{\tilde{q}^2\tilde{t}} du\, e^u\right.$$

$$\left.+(\tilde{q}^2\tilde{t}-1)\frac{1}{\tilde{q}^2}\int_0^{\tilde{q}^2\tilde{t}} du \exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right]\right\}$$

$$+O(\tilde{q}^2). \tag{E29}$$

To find the asymptotic behavior of the last term we write

$$\frac{1}{\tilde{q}^2}\int_0^y du \exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right] \tag{E30}$$

$$=\frac{1}{\tilde{q}^2}\int_0^y du\left\{\exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right]-\int_a\frac{1}{\frac{4u}{\tilde{q}^2}+2a}\right\} \tag{E31}$$

$$+\frac{1}{\tilde{q}^2}\int_0^y du\int_a\frac{1}{\frac{4u}{\tilde{q}^2}+2a}. \tag{E32}$$

In the integral on the second line, we can take the limit $\tilde{q}\to 0$ by making the change of variable $\lambda=u/\tilde{q}^2$, and the second can be done exactly. We thus have

$$\frac{1}{\tilde{q}^2}\int_0^{\tilde{q}^2\tilde{t}} du \exp\left[-\int_a \ln\left(\frac{4u}{\tilde{q}^2}+2a\right)\right]$$

$$=\frac{1}{4}\ln\frac{y}{\tilde{q}^2}-\int_a \ln\frac{a}{2}+\int_0^\infty d\lambda \exp\left(-\int_a \ln(4\lambda+2a)\right)$$

$$-\int_a\frac{1}{4\lambda+2a}+O(\tilde{q}^2).$$

Finally, using

$$\int_0^y du\,\frac{e^u-1}{4u}=\frac{1}{4}[-\gamma_E+\mathrm{Ei}(y)-\ln y] \tag{E33}$$

one has, up to terms of order $\tilde{q}^2$,

$$\mathcal{G}_{\tilde{t}}^{\tilde{q}}=1+B^*[(y-1)\mathrm{Ei}(y)+1-e^y+(1-y)(\ln\tilde{q}^2+\rho)],$$

$$\rho=\gamma_E+\int_a \ln\frac{a}{2}-4\int_0^\infty d\lambda\,\exp\left(-\int_a \ln(4\lambda+2a)\right)$$
$$+4\int_a\frac{1}{4\lambda+2a}, \tag{E34}$$

which yields the scaling function given in the text.

APPENDIX F: NONEQUILIBRIUM DYNAMICS OF THE CO MODEL

1. Some useful expressions

To begin with, we give the explicit expression of $\Delta_{ltt'}$ and $\Sigma_{ltt'}$ and their limiting expression when $l\to\infty$ in the case of nonequilibrium dynamics. The general expression of $\Delta_{ltt'}(u)$, i.e., the first line of Eq. (E5) is still valid for nonequilibrium dynamics. To evaluate it, we only need the expression of $C_{ltt'}^{x=0}$ that we compute from Eq. (122) using the same cutoff function $c(z)$ as previously (17),

$$C_{ltt'}^{x=0}=\frac{T}{4\pi}\int_a\left[\ln\left(t+t'+\frac{a}{2\Lambda_0^2}\right)-\ln\left(|t-t'|+\frac{a}{2\Lambda_0^2}\right)\right.$$
$$\left.-\frac{T}{4\pi}\ln\left(t+t'+\frac{a}{2\Lambda_l^2}\right)+\ln\left(|t-t'|+\frac{a}{2\Lambda_l^2}\right)\right]. \tag{F1}$$

Notice that the response function $R_{tt'}^{x=0}$ has its equilibrium expression. From Eqs. (E5) and (F1), one obtains

$$\Delta_{ltt'}(u)=e^{-1/2C_{ltt}^{x=0}-1/2C_{lt't'}^{x=0}+C_{ltt'}^{x=0}}\Delta_{l=0tt'}(u). \tag{F2}$$

Using Eq. (F1) one has, using $T=T_c=4\pi$ to this order,

$$\lim_{l\to\infty}-\frac{1}{2}C_{ltt}^{x=0}-\frac{1}{2}C_{lt't'}^{x=0}+C_{ltt'}^{x=0}$$
$$=\int_a-\ln\left(\Lambda_0^2|t-t'|+\frac{a}{2}\right)+\ln\left(\Lambda_0^2(t+t')+\frac{a}{2}\right)$$
$$-\frac{1}{2}\ln\left(\Lambda_0^2 t+\frac{a}{4}\right)-\frac{1}{2}\ln\left(\Lambda_0^2 t'+\frac{a}{4}\right)+\ln\frac{a}{4} \tag{F3}$$

and using the definition (88), one obtains finally

$$D_{tt'}=\lim_{l\to\infty}D_{ltt'}=\frac{\Lambda_0^2T_cB^*}{2}\exp\left\{\int_a-\ln\left(\Lambda_0^2|t-t'|+\frac{a}{2}\right)\right.$$
$$\left.+\ln\left[\Lambda_0^2(t+t')+\frac{a}{2}\right]\right\}\exp\left[\int_a-\frac{1}{2}\ln\left(\Lambda_0^2 t+\frac{a}{4}\right)\right.$$
$$\left.-\frac{1}{2}\ln\left(\Lambda_0^2 t'+\frac{a}{4}\right)\right], \tag{F4}$$

where we have used the expression of B^* given in Eq. (95). The expression for $\Sigma_{ltt'}$ can be obtained in a very similar way,

$$\Sigma_{tt'}=\lim_{l\to\infty}\Sigma_{ltt'}=\frac{-\Lambda_0^4B^*}{2}\int_a\left\{\frac{\theta(t-t')}{\Lambda_0^2(t-t')+\frac{a}{2}}\exp\left[\int_a-\ln\left(\Lambda_0^2|t-t'|+\frac{a}{2}\right)\right]\exp\left\{\int_a\ln\left[\Lambda_0^2(t+t')+\frac{a}{2}\right]-\frac{1}{2}\ln\left(\Lambda_0^2 t+\frac{a}{4}\right)\right.\right.$$
$$\left.-\frac{1}{2}\ln\left(\Lambda_0^2 t'+\frac{a}{4}\right)\right\}-\delta(t-t')\int_0^t dt_1\frac{1}{\Lambda_0^2(t-t_1)+\frac{a}{2}}\exp\left[\int_a-\ln\left(\Lambda_0^2|t-t_1|+\frac{a}{2}\right)\right]$$
$$\left.\times\exp\left[\int_a\ln[\Lambda_0^2(t+t_1)+a/2]-\frac{1}{2}\ln\left(\Lambda_0^2 t+\frac{a}{4}\right)-\frac{1}{2}\ln\left(\Lambda_0^2 t_1+\frac{a}{4}\right)\right]\right\}. \tag{F5}$$

These expressions (F4),(F5) will be very useful to determine explicit expressions for $\mathcal{R}^q_{tt'}=\lim_{l\to\infty}\mathcal{R}^q_{ltt'}$ and $\mathcal{C}^q_{tt'}=\lim_{l\to\infty}\mathcal{C}^q_{ltt'}$ by solving perturbatively Eqs. (99) and (100).

2. Nonequilibrium response function: detailed calculations

The starting point of our analysis is Eq. (99) that we solve perturbatively by replacing, in the right hand side (rhs) of this equation, $\mathcal{R}^q_{ltt'}$ by its bare value. One obtains, in the limit $l\to\infty$, using Eq. (F5), and in terms of the rescaled variables $\tilde t=\Lambda_0^2 t$, $\tilde q=q/\Lambda_0$,

$$\partial_{\tilde t}\mathcal{R}^{\tilde q}_{\tilde t\tilde t'}+\tilde q^2\mathcal{R}^{\tilde q}_{\tilde t\tilde t'}=\frac{B^*}{2}\int_0^{\tilde t}dt_1\frac{1}{(\tilde t-t_1)+\frac{a}{2}}\exp\left[\int_a -\ln\left(\tilde t-t_1+\frac{a}{2}\right)\right]\times\exp\left\{\int_a \ln\left[(\tilde t+t_1)+\frac{a}{2}\right]-\frac{1}{2}\ln\left(\tilde t+\frac{a}{4}\right)-\frac{1}{2}\ln\left(t_1+\frac{a}{4}\right)\right\}[\theta(t_1-\tilde t')\times e^{-\tilde q^2(t_1-\tilde t')}-e^{-\tilde q^2(\tilde t-\tilde t')}]. \tag{F6}$$

Let us first focus on the last term in the rhs of Eq. (F6), where we make the change of variable $t_1=u\tilde t$ and analyzing the limit $\tilde t\gg 1$,

$$-\frac{B^*}{2}e^{-\tilde q^2(\tilde t-\tilde t')}\int_0^1 du\int_a\frac{1}{1-u+\frac{a}{2\tilde t}}\times\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)\right]\exp\left[\int_a\ln\left(1+u+\frac{a}{2\tilde t}\right)-\frac{1}{2}\ln\left(u+\frac{a}{4\tilde t}\right)-\frac{1}{2}\ln\left(\tilde t+\frac{a}{4}\right)-\frac{1}{2}\ln(\tilde t)\right]$$
$$\sim-\frac{B^*}{2}e^{-\tilde q^2(\tilde t-\tilde t')}\int_0^1\frac{du}{\tilde t}\int_a\frac{1}{1-u+\frac{a}{2\tilde t}}\times\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)\right]\times\exp\left[\int_a\ln\left(1+u+\frac{a}{2\tilde t}\right)-\frac{1}{2}\ln\left(u+\frac{a}{4\tilde t}\right)\right]=Q. \tag{F7}$$

In the integrand one cannot directly take the limit $\tilde t\to\infty$ because it generates a divergence of the integral when $u\to 1$. Therefore we subtract the divergent term in the following way:

$$Q=-\frac{B^*}{2}e^{-\tilde q^2(\tilde t-\tilde t')}\int_0^1\frac{du}{\tilde t}\int_a\frac{2}{1-u+\frac{a}{2\tilde t}}\times\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)\right]-\frac{B^*}{2}e^{-\tilde q^2(\tilde t-\tilde t')}\times\int_0^1\frac{du}{\tilde t}\int_a\frac{1}{1-u+\frac{a}{2\tilde t}}\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)\right]\times\left\{\exp\left[\int_a\ln\left(1+u+\frac{a}{2\tilde t}\right)-\frac{1}{2}\ln\left(u+\frac{a}{4\tilde t}\right)\right]-2\right\}. \tag{F8}$$

Interestingly, except in the first line, one can take directly the limit $\tilde t\to\infty$ in the integrand of the two last lines using

$$\lim_{\tilde t\to\infty}\frac{1}{1-u+\frac{a}{2\tilde t}}\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)\right]\times\left\{\exp\left[\int_a\ln\left(1+u+\frac{a}{2\tilde t}\right)-\frac{1}{2}\ln\left(u+\frac{a}{4\tilde t}\right)\right]-2\right\}=\frac{1}{(1-u)^2}\left(\frac{1+u}{\sqrt{u}}-2\right)=\frac{1}{\sqrt{u}(1+\sqrt{u})^2} \tag{F9}$$

and the divergence for $u\to 1$ is cured. Then all the remaining integrals can be performed exactly, giving finally

$$Q=B^*e^{-\tilde q^2(\tilde t-\tilde t')}\left[-\exp\left(-\int_a\ln\frac{a}{2}\right)+\frac{1}{2\tilde t}+O(\tilde t^{-2})\right]. \tag{F10}$$

We now perform exactly the same manipulations on the first term in the rhs of Eq. (F6). Performing the change of variable $t_1=u\tilde t$ and considering the limit $\tilde t\gg 1$ (keeping $\tilde q^2\tilde t$, $\tilde q^2\tilde t'$ and $\tilde t/\tilde t'$ fixed), one obtains

$$\frac{B^*}{2}\int_{\tilde t'/\tilde t}^1\frac{du}{\tilde t}\int_a\frac{e^{-\tilde q^2(u\tilde t-\tilde t')}}{1-u+\frac{a}{2\tilde t}}\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)+\ln\left(1+u+\frac{a}{2\tilde t}\right)-\frac{1}{2}\ln\left(u+\frac{a}{4\tilde t}\right)\right]$$
$$=B^*\int_{\tilde t'/\tilde t}^1\frac{du}{\tilde t}\int_a\frac{e^{-\tilde q^2(u\tilde t-\tilde t')}}{1-u+\frac{a}{2\tilde t}}\exp\left[\int_a -\ln\left(1-u+\frac{a}{2\tilde t}\right)\right]+\frac{B^*}{2}\int_{\tilde t'/\tilde t}^1\frac{du}{\tilde t}\frac{e^{-\tilde q^2(u\tilde t-\tilde t')}}{\sqrt{u}(1+\sqrt{u})^2}+O(\tilde t^{-2}), \tag{F11}$$

where we have used the same trick (F9) as previously. Using Eq. (F11) together with Eq. (F10), one can write Eq. (F6) in a rather simple way

$$\partial_{\tilde{t}}\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}+\tilde{q}^2\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=4B^*\int_{\tilde{t}'}^{\tilde{t}}dt_1\int_a\frac{1}{t-t_1+\frac{a}{2}}\exp\left(-\int_a \ln[4(\tilde{t}-t_1)+2a]\right)\times e^{-\tilde{q}^2(t_1-\tilde{t}')}-B^*e^{-\tilde{q}^2(\tilde{t}-\tilde{t}')}\exp\left[-\int_a\ln\left(\frac{a}{2}\right)\right]+\frac{B^*}{2}\left(\int_{\tilde{t}'}^{\tilde{t}}\frac{dt_1}{\sqrt{\tilde{t}t_1}}\frac{e^{-\tilde{q}^2(t_1-\tilde{t}')}}{(\sqrt{\tilde{t}}+\sqrt{t_1})^2}+\frac{e^{-\tilde{q}^2(\tilde{t}-\tilde{t}')}}{\tilde{t}}\right). \tag{F12}$$

The two first lines correspond to equilibrium fluctuations (E23) and their contribution to the response function has already been computed (105). The last term does not depend anymore on the cutoff function and characterizes the contributions coming from nonequilibrium fluctuations. The linearity of this equation suggests then to look for a solution under the form $\mathcal{R}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\mathcal{R}^{\tilde{q}\,\mathrm{eq}}_{\tilde{t}\tilde{t}'}+\mathcal{R}^{\tilde{q}\,\mathrm{noneq}}_{\tilde{t}\tilde{t}'}$, where $\mathcal{R}^{\tilde{q}\,\mathrm{eq}}_{\tilde{t}\tilde{t}'}=\tilde{q}^{z-2}F_R^{\mathrm{eq}}[\tilde{q}^z(\tilde{t}-\tilde{t}')]$ (103) and $\mathcal{R}^{\tilde{q}\,\mathrm{noneq}}_{\tilde{t}\tilde{t}'}=e^{-\tilde{q}^2(\tilde{t}-\tilde{t}')}H^{\tilde{q}}_{\tilde{t}\tilde{t}'}$, with $H^{\tilde{q}}_{\tilde{t}\tilde{t}'}$ determined by Eq. (F6),

$$\partial_{\tilde{t}}H^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\frac{B^*}{2}\left(\int_{\tilde{t}'}^{\tilde{t}}\frac{dt_1}{\sqrt{\tilde{t}t_1}}\frac{e^{-\tilde{q}^2(t_1-\tilde{t})}}{(\sqrt{\tilde{t}}+\sqrt{t_1})^2}+\frac{1}{\tilde{t}}\right),$$

$$H^{\tilde{q}}_{\tilde{t}\tilde{t}}=H^{\tilde{q}}_{\tilde{t}\tilde{t}^-}=0. \tag{F13}$$

This allows to write a close expression for the perturbative expansion of $\mathcal{R}^{\tilde{q}\,\mathrm{noneq}}_{\tilde{t}\tilde{t}'}$ in terms of the scaling variables $v'=\tilde{q}^2(\tilde{t}-\tilde{t}')$, $u=\tilde{t}/\tilde{t}'$,

$$\mathcal{R}^{\tilde{q}\,\mathrm{noneq}}_{\tilde{t}\tilde{t}'}=\frac{B^*e^{-v'}}{2}\left(\int_{v'/(u-1)}^{uv'/(u-1)}\frac{dt_2}{\sqrt{t_2}}\int_{v'/(u-1)}^{t_2}\frac{dt_1}{\sqrt{t_1}}\times\frac{e^{t_2-t_1}}{(\sqrt{t_1}+\sqrt{t_2})^2}+\ln u\right). \tag{F14}$$

Unfortunately, it is quite difficult to extract directly the asymptotic behaviors from this double integral. However, one can perform straightforward (although tedious) manipulations to obtain a quasiexplicit expression for $\mathcal{R}^{\tilde{q}\,\mathrm{noneq}}_{\tilde{t}\tilde{t}'}$. Performing the natural change of variables $\alpha=\sqrt{t_2}-\sqrt{t_1}$, $\beta=\sqrt{t_2}+\sqrt{t_1}$ one is left with integrals over one variable,

$$\frac{B^*}{2}e^{-v}\int_{v/(u-1)}^{uv/(u-1)}\frac{dt_2}{\sqrt{t_2}}\int_{v/(u-1)}^{t_2}\frac{dt_1}{\sqrt{t_1}}\frac{e^{t_2-t_1}}{(\sqrt{t_1}+\sqrt{t_2})^2}=B^*e^{-v}\left[\frac{u-1}{8vu}-\frac{u-1}{8v}+\mathcal{Q}\left(\frac{v}{u-1},u\right)+\mathcal{Q}\left(\frac{-vu}{u-1},\frac{1}{u}\right)\right],$$

$$\mathcal{Q}(x,y)=\frac{1}{x}\int_1^{\sqrt{y}}d\beta\frac{e^{x(\beta^2-1)}}{(\beta+1)^3}. \tag{F15}$$

Performing further manipulations we find that one can write

$$\mathcal{R}^{\tilde{q}\,\mathrm{noneq}}_{\tilde{t}\tilde{t}'}=\theta\ln u e^{-v}+\tau F_R^{1\,\mathrm{noneq}}(v',u)+O(\tau^2), \tag{F16}$$

$$\theta=B^*+O(\tau^2), \tag{F17}$$

where the logarithmic behavior determining θ has been extracted such that the function $F_R^{1\mathrm{noneq}}(v,u)$ has a good limit for $u\to\infty$, as will be shown below. A useful expression for this function is found as

$$F_R^{1\mathrm{noneq}}(v,u)=e^{\gamma_E}\Bigg\{1-e^{-v}-\sqrt{\pi}\sqrt{\frac{vu}{u-1}}e^{v/(u-1)}\left(\mathrm{erf}\sqrt{\frac{vu}{u-1}}-\mathrm{erf}\sqrt{\frac{v}{u-1}}\right)-\sqrt{\pi}\sqrt{\frac{v}{u-1}}e^{-vu/(u-1)}\left(\mathrm{erfi}\sqrt{\frac{vu}{u-1}}-\mathrm{erfi}\sqrt{\frac{v}{u-1}}\right)+e^{-v}(1-v)[\mathrm{Ei}(v)-\ln v-\gamma_E]+2e^{-v}\frac{vu}{u-1}\left[\frac{1}{u}\left(\frac{v}{u-1}+\frac{1}{2}\right){}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},-\frac{v}{u-1}\right)-\left(-\frac{vu}{u-1}+\frac{1}{2}\right){}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},\frac{vu}{u-1}\right)\right]+e^{-v}\pi\left(\frac{1}{2}-\frac{vu}{u-1}\right)\mathrm{erf}\sqrt{\frac{v}{u-1}}\mathrm{erfi}\sqrt{\frac{vu}{u-1}}-2\sqrt{\pi}(1-v)\frac{e^{-v}}{\sqrt{u}}\int_0^{\sqrt{vu/(u-1)}}ds\,e^{-s^2/u}\mathrm{erfi}(s)+2(1-v)e^{-v}\ln\frac{1+\frac{1}{\sqrt{u}}}{2}+\frac{ve^{-v}}{u-1}\ln u\Bigg\}, \tag{F18}$$

where erf z is the error function, erfi z is the imaginary error function:

$$\text{erf}\, z = \frac{2}{\sqrt{\pi}}\int_0^z ds e^{-s^2}, \tag{F19}$$

$$\text{erf}i\, z = \frac{2}{\sqrt{\pi}}\int_0^z ds e^{s^2} \tag{F20}$$

with erfi $z = -i$ erfi z. One has the following asymptotic behaviors:

$$\text{erf}\, z \sim 2z/\sqrt{\pi}, \quad z \ll 1, \tag{F21}$$

$$\text{erf}\, z \sim 1 - e^{-z^2}/(\sqrt{\pi}z), \quad z \gg 1 \tag{F22}$$

and

$$\text{erfi}\, z \sim 2z/\sqrt{\pi}, \quad z \ll 1, \tag{F23}$$

$$\text{erfi}\, z \sim e^{z^2}/(\sqrt{\pi}z), \quad z \gg 1, \tag{F24}$$

and ${}_2F_2(\{1,1\},\{\frac{3}{2},2\},z)$ is a generalized hypergeometric series which has the following asymptotic behaviors:

$${}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},z\right) \sim 1 + O(z), \tag{F25}$$

$${}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},z\right) \sim_{z\to+\infty} \frac{\sqrt{\pi}}{2}\frac{e^z}{z^{3/2}}[1+O(z^{-1})],$$

$${}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},z\right) \sim_{z\to-\infty} -\frac{\ln(-z)}{2z}. \tag{F26}$$

Under this form, asymptotic behaviors are more easily obtained. Note that we have also performed numerical checks that Eq. (F18) and the starting integral (F14) do indeed coincide.

Note some simple formulas for the same point response:

$$\mathcal{R}_{\tilde{t}\tilde{t}'}^{x=0} = \frac{1}{2\pi z(t-t')} h(t/t'), \tag{F27}$$

$$h(u) = u^\theta \int_0^\infty dv F_R(v,u). \tag{F28}$$

a. Expansion at large u, v fixed

The asymptotic behavior of $F_R^1(v,u)$ is easily obtained in this limit. From Eq. (123), one has $\lim_{u\to\infty} F_R^1(v,u) = F_R^{1\text{eq}}(v) + \lim_{u\to\infty} F_R^{1\text{noneq}}(v,u)$, where $F_R^{1\text{eq}}(v)$ is given in Eq. (104). On the expressions (F16), (F18) together with the asymptotic behaviors (F21), (F23), (F25) we see that all terms vanish in this limit except the following ones:

$$\begin{aligned}&\lim_{u\to\infty, v\,\text{fixed}} F_R^{1\text{noneq}}(v,u)\\ &= -F_R^{1\text{eq}}(v) + e^{\gamma_E}\Bigg\{ -\sqrt{\pi v}\,\text{erf}\sqrt{v} - e^{-v}\\ &\times\left[(1-v)\ln(4ve^{\gamma_E}) - 2v\left(v-\frac{1}{2}\right)\right.\\ &\left.\times {}_2F_2\left(\{1,1\},\left\{\frac{3}{2},2\right\},v\right)\right]\Bigg\},\end{aligned}$$

which leads to Eq. (125) in the text.

b. Expansion at large v, u fixed

Although one can extract more rigorously the large v behavior at u fixed from the complete expression (F16), it is easier to compute it from the starting integral in Eq. (F14). Indeed, in the large v limit, the integral will be dominated by the region $t_2 - t_1 \sim v$, i.e., one can replace in the integrand (except of course in the term $e^{t_2-t_1}$) t_2 by $vu/(u-1)$ and t_1 by $v/(u-1)$,

$$\begin{aligned}&\frac{B^* e^{-v}}{2}\int_{v/(u-1)}^{uv/(u-1)} \frac{dt_2}{\sqrt{t_2}}\int_{v/(u-1)}^{t_2}\frac{dt_1}{\sqrt{t_1}}\frac{e^{t_2-t_1}}{(\sqrt{t_1}+\sqrt{t_2})^2}\\ &\sim \frac{B^*}{2}\frac{1}{v^2}\frac{(\sqrt{u}-1)^2}{\sqrt{u}}e^{-v}\\ &\times\int_{v/(u-1)}^{vu/(u-1)} dt_2 e^{t_2}\int_{(v/(u-1)}^{vu/(u-1)} dt_1 e^{-t_1},\end{aligned} \tag{F29}$$

which leads finally to

$$F_R^{1\text{noneq}}(v,u) \sim \frac{B^*}{2\tau}\frac{1}{v^2}\frac{(\sqrt{u}-1)^2}{\sqrt{u}} + O(v^{-3}). \tag{F30}$$

We have checked that we obtain the same result by performing this expansion on Eq. (F16). Finally, using the large v behavior of $F_R^{1\text{eq}}$ (107) and the value of B^* (95), one obtains

$$F_R^1(v,u) \sim e^{\gamma_E}\frac{1}{2v^2}\frac{u+1}{\sqrt{u}} + O(v^{-3}), \tag{F31}$$

which gives Eq. (127) in the text.

c. The limit of vanishing momentum

The limit $\tilde{q}\to 0$ is easily obtained by looking for the leading term in $F_R(v,u)$ when $v = \tilde{q}^2(\tilde{t}-\tilde{t}')\to 0$. Using Eq. (F18) together with Eqs (123), and (104), one has

$$F_R^1(v) \sim -e^{\gamma_E}\left(\ln v + \gamma_E - 2\ln\frac{1+\frac{1}{\sqrt{u}}}{2}\right). \tag{F32}$$

This logarithmic behavior together with Eq. (98) cancels the $\log \tilde{q}$ divergence in Eq. (124) and allows to take the limit of vanishing momentum. We also give here the expression of $F_R^{1\text{noneq}}(0,u)$, obtained from Eq. (F16),

$$F_R^{1\text{noneq}}(0,u)=2e^{\gamma_E}\ln\frac{1+\frac{1}{\sqrt{u}}}{2}. \tag{F33}$$

This will be useful for further applications.

3. Nonequilibrium correlation function: Detailed calculations

The starting point of our analysis is the following expression given in the text (129), for $\tilde{t}>\tilde{t}'$:

$$\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}=\lim_{l\to\infty}\mathcal{C}^{\tilde{q}}_{l\tilde{t}\tilde{t}'}, \tag{F34}$$

$$=2T\int_0^{\tilde{t}'}dt_1\mathcal{R}^{\tilde{q}}_{\tilde{t}t_1}\mathcal{R}^{\tilde{q}}_{\tilde{t}'t_1}+\int_0^{\tilde{t}}dt_1\int_0^{\tilde{t}'}dt_2\mathcal{R}^{\tilde{q}}_{\tilde{t}t_1}D_{t_1t_2}\mathcal{R}^{\tilde{q}}_{\tilde{t}'t_2}, \tag{F35}$$

where $D_{t_1t_2}=\lim_{l\to\infty}D_{lt_1t_2}$ is given in Eq. (F4), that we expand perturbatively using the expression we obtained for $\mathcal{R}^q_{tt'}$ (124). As we did previously for the response function, we could keep the complete cutoff dependence in Eq. (F4). However, given the complexity of these manipulations and the experience we acquired before, we know that the only cutoff dependence is contained in an overall nonuniversal scale $\tilde{q}\to\lambda\tilde{q}$. For these reasons we will perform the computation using a simplified cutoff $\hat{c}(a)=\delta(a-a_0)$ and we will choose $a_0=2$ for simplicity. $D_{t_1t_2}$ can then be written as (F4)

$$D_{t_1t_2}=\frac{1}{2}e^{\gamma_E}T_c\tau\frac{t_1+t_2}{(|t_1-t_2|+1)\sqrt{t_1t_2}}+O(\tau^2), \tag{F36}$$

where we have dropped the a_0 dependence where it turns out to be unimportant.

Performing the integrals that do not involve $F_R^1(v,u)$, one has

$$\begin{aligned}\mathcal{C}^q_{tt'}=&\frac{T}{q^2}F_C^0(v,u)+\frac{T}{q^2}\theta\ln uF_C^0(v,u)+\frac{T}{q^2}(z-2)\ln q\left(v\frac{\partial F_C^0(v,u)}{\partial v}+F_C^0(v,u)\right)+\frac{2T\theta}{q^2}e^{-v(1+u)/(u-1)}\\&\times\left[\mathrm{Ei}\left(\frac{2v}{u-1}\right)-\ln\left(\frac{2v}{u-1}\right)-\gamma_E\right]+\frac{2T\tau}{q^2}\frac{v}{u-1}\int_0^1 ds F_R^0\left(\frac{u-s}{u-1}v\right)F_R^1\left(\frac{1-s}{u-1}v,\frac{1}{s}\right)+F_R^0\left(\frac{1-s}{u-1}v\right)F_R^1\left(\frac{u-s}{u-1}v,\frac{u}{s}\right)\\&+\frac{e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(u-1)}\int_0^{uv/(u-1)}dt_1\int_0^{v/(u-1)}dt_2e^{(t_1+t_2)}\left(\frac{1}{|t_1-t_2|+q^2}+\frac{1}{2}\frac{\sqrt{t_1}-\sqrt{t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}\right),\end{aligned} \tag{F37}$$

where we have used the trick (F9), and dropped the prime in $v'=\tilde{q}^2(\tilde{t}-\tilde{t}')$ for simplicity. A natural way to perform this computation is to use for $F_R^1(v,u)$ the decomposition in an equilibrium and a nonequilibrium contributions (123). Parts of Eq. (F37) can then be computed analytically,

$$\begin{aligned}&\frac{2T\tau}{q^2}\frac{v}{u-1}\int_0^1 dsF_R^0\left(\frac{u-s}{u-1}v\right)F_R^{1\text{eq}}\left(\frac{1-s}{u-1}v\right)\\&\quad+F_R^0\left(\frac{1-s}{u-1}v\right)F_R^{1\text{eq}}\left(\frac{u-s}{u-1}v\right)\\&\quad+\frac{e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(u-1)}\int_0^{uv/(u-1)}dt_1\int_0^{v/(u-1)}dt_2\\&\quad\times\frac{e^{(t_1+t_2)}}{|t_1-t_2|+q^2}\\&=-\frac{e^{\gamma_E}\tau}{q^2}\ln(q^2e^{\gamma_E})TF_C^0(v,u)\\&\quad+\frac{1}{2}e^{\gamma_E}\tau T\left\{\frac{e^{-[v(u+1)]/(u-1)}}{q^2}\left[-4-2\frac{uv}{u-1}\mathrm{Ei}\left(\frac{uv}{u-1}\right)\right.\right.\\&\quad\left.-2\frac{v}{u-1}\mathrm{Ei}\left(\frac{v}{u-1}\right)\right]+\frac{2}{q^2}(e^{-vu/(u-1)}+e^{-v/(u-1)})\\&\quad\left.+\frac{2}{q^2}[e^{-v}-1+ve^{-v}\mathrm{Ei}(v)]\right\}.\end{aligned} \tag{F38}$$

The expressions (F37) together with Eq. (F38) allow one to identify the following perturbative scaling behavior (115):

$$\begin{aligned}\mathcal{C}^q_{tt'}=&\frac{T}{q^2}\left(F_C^0(v,u)+(z-2)v\ln q\frac{\partial F_C^0(v,u)}{\partial v}\right.\\&\left.+\theta\ln uF_C^0(v,u)+\tau F_C^1(v,u)\right)+O(\tau^2),\end{aligned} \tag{F39}$$

$$F_C^1(v,u)=2e^{\gamma_E}e^{-v(1+u)/(u-1)}\left[\mathrm{Ei}\left(\frac{2v}{u-1}\right)-\ln\left(\frac{2v}{u-1}\right)-\gamma_E\right]+\frac{1}{2}e^{\gamma_E}\Bigg\{e^{-v(u+1)/(u-1)}$$
$$\times\left[-4-2\frac{uv}{u-1}\mathrm{Ei}\left(\frac{uv}{u-1}\right)-2\frac{v}{u-1}\mathrm{Ei}\left(\frac{v}{u-1}\right)\right]+2(e^{-vu/(u-1)}+e^{-v/(u-1)})+2[e^{-v}-1+ve^{-v}\mathrm{Ei}(v)]\Bigg\}$$
$$+2\frac{v}{u-1}\int_0^1 ds\left\{F_R^0\left(\frac{u-s}{u-1}v\right)F_R^{1\mathrm{noneq}}\left(\frac{1-s}{u-1}v,\frac{1}{s}\right)+F_R^0\left(\frac{1-s}{u-1}v\right)F_R^{1\mathrm{noneq}}\left(\frac{u-s}{u-1}v,\frac{u}{s}\right)\right\}$$
$$+\frac{1}{2}e^{\gamma_E}e^{-v(1+u)/(u-1)}\int_0^{vu/(u-1)}dt_1\int_0^{v/(u-1)}dt_2\frac{(\sqrt{t_1}-\sqrt{t_2})e^{t_1+t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}$$

with the exponents z and θ given in Eqs. (98) and (123). The scaling functions are universal up to a cutoff dependent additive constant. It was explicitly computed for the equilibrium response in Eq. (E34). Here, we do not determine it and thus we can drop some multiplicative factors of momentum in the $\ln\tilde{q}$ term.

a. Expansion at large u, v fixed

First, one has

$$F_C^0(v,u)=e^{-v}-e^{-v(u+1)/(u-1)}\sim\frac{2e^{-v}v}{u}+O(u^{-2}).\tag{F40}$$

We now focus on the asymptotic behavior of $F_C^1(v,u)$ for large u, keeping v fixed. Using the small argument behavior of $\mathrm{Ei}(z)\sim\ln z+\gamma_E+O(z)$, one has for the first line of Eq. (F39) in this limit

$$2e^{\gamma_E}e^{-v(1+u)/(u-1)}\left[\mathrm{Ei}\left(\frac{2v}{u-1}\right)-\ln\left(\frac{2v}{u-1}\right)-\gamma_E\right]\sim O(u^{-1}).\tag{F41}$$

Again using the small argument behavior of $\mathrm{Ei}(z)$, one has

$$+\frac{1}{2}e^{\gamma_E}\Bigg\{e^{-[v(u+1)]/(u-1)}\left[-4-2\frac{uv}{u-1}\mathrm{Ei}\left(\frac{uv}{u-1}\right)-2\frac{v}{u-1}\mathrm{Ei}\left(\frac{v}{u-1}\right)\right]+2(e^{-vu/(u-1)}+e^{-v/(u-1)})+2[e^{-v}-1+ve^{-v}\mathrm{Ei}(v)]\Bigg\}\sim e^{\gamma_E}e^{-v}v\frac{\ln u}{u}+O(u^{-1}).\tag{F42}$$

One then analyzes the integrals involving $F_R^{1\mathrm{noneq}}(v,u)$,

$$\frac{2T\tau}{q^2}\frac{v}{u-1}\int_0^1 ds\left\{F_R^0\left(\frac{u-s}{u-1}v\right)F_R^{1\mathrm{noneq}}\left(\frac{1-s}{u-1}v,\frac{1}{s}\right)+F_R^0\left(\frac{1-s}{u-1}v\right)F_R^{1\mathrm{noneq}}\left(\frac{u-s}{u-1}v,\frac{u}{s}\right)\right\}\tag{F43}$$

$$\sim\frac{2T\tau}{q^2}\frac{1}{u}\left\{vF_R^0(v)\int_0^1 dsF_R^{1\mathrm{noneq}}\left(0,\frac{1}{s}\right)+vF_R^0(0)F_R^{1\mathrm{noneq}}(v,\infty)+O(u^{-1})\right\}\tag{F44}$$

and the remaining integral in Eq. (F39) where we perform the natural change of variable $\alpha=\sqrt{t_1}$, $\beta=\sqrt{t_2}$,

$$\frac{e^{\gamma_E}T_c\tau}{2q^2}e^{-v(1+u)/(u-1)}\times\int_0^{vu/(u-1)}dt_1\int_0^{v/(u-1)}dt_2\frac{(\sqrt{t_1}-\sqrt{t_2})e^{t_1+t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}$$
$$=\frac{2e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(u-1)}\int_{\sqrt{v/(u-1)}}^{\sqrt{vu/(u-1)}}d\alpha\times\int_0^{\sqrt{v/(u-1)}}d\beta e^{\alpha^2+\beta^2}-\frac{4e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(1-u)}\times\int_{\sqrt{v/(u-1)}}^{\sqrt{vu/(u-1)}}d\alpha\int_0^{\sqrt{}}v/(u-1)d\beta e^{\alpha^2+\beta^2}\frac{\beta}{\alpha+\beta}.$$

The first double integral can be performed exactly,

$$\frac{2e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(1-u)}\int_{\sqrt{v/(u-1)}}^{\sqrt{vu/(u-1)}}d\alpha\int_0^{\sqrt{v/(u-1)}}d\beta e^{\alpha^2+\beta^2}$$
$$=\frac{\pi e^{\gamma_E}T_c\tau}{2q^2}e^{-v(1+u)/(u-1)}\times\left(\mathrm{erfi}\sqrt{\frac{uv}{u-1}}-\mathrm{erf}i\sqrt{\frac{v}{u-1}}\right)\mathrm{erfi}\sqrt{\frac{v}{u-1}}$$
$$\sim\frac{\sqrt{\pi}e^{\gamma_E}T_c\tau}{q^2}e^{-v}\sqrt{\frac{v}{u}}\mathrm{erfi}\sqrt{v}+O(u^{-1}).\tag{F45}$$

And we expand the second one in the following way:

$$-\frac{4e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(u-1)}\int_{\sqrt{v/(u-1)}}^{\sqrt{vu/(u-1)}}d\alpha \times\int_0^{\sqrt{v/(u-1)}}d\beta e^{\alpha^2+\beta^2}\frac{\beta}{\alpha+\beta}$$

$$=-2\frac{2e^{\gamma_E}T_c\tau}{q^2}e^{-v(1+u)/(u-1)}\int_{\sqrt{v/(u-1)}}^{\sqrt{vu/(u-1)}}d\alpha e^{\alpha^2}\times\left[\frac{1}{2\alpha}\frac{v}{u}+\sum_{n>2}\left(\frac{v}{u}\right)^{n/2}\frac{a_n}{\alpha^{n-1}}[1+O(\alpha)]\right]$$

$$\sim-\frac{2e^{\gamma_E}T_c\tau}{q^2}e^{-v}\frac{v}{u}\int_{\sqrt{v/u}}^{\sqrt{v}}\frac{d\alpha}{\alpha}e^{\alpha^2}+O(u^{-1})$$

$$\sim-\frac{e^{\gamma_E T_c}\tau}{q^2}e^{-v}\frac{v}{u}\ln u+O(u^{-1}). \tag{F46}$$

Finally, Eqs. (F40), (F42), (F45), and (F46) lead to the asymptotic following form for $F_C(v,u)$ in the limit $u\to\infty$, v fixed,

$$\lim_{v\to\infty}F_C(v,u)=\frac{2ve^{-v}}{u}+\tau\frac{F^1_{C,\infty}(v)}{\sqrt{u}}+O(u^{-2},\tau u^{-1},\tau^2),$$

$$F^1_{C,\infty}(v)=e^{\gamma_E}e^{-v}\sqrt{\pi v}\,\mathrm{erfi}\sqrt{v}, \tag{F47}$$

notice that the subdominant terms in $\ln u/u$ cancel between Eqs. (F42) and (F46) so that the leading corrections are of order u^{-1}. Equation (F47) gives the asymptotic behavior given in the text (132).

b. Expansion at large v, u fixed

In this limit, the terms in the four first lines of Eq. (F39) decay exponentially in this limit. The fifth line, however (which corresponds to the equilibrium contribution), decays like a power law. Indeed, using the large v behavior of $\mathrm{Ei}(v)\sim e^v[1/v+1/v^2+0(v^{-3})]$ one has

$$e^{\gamma_E}[e^{-v}-1+ve^{-v}\mathrm{Ei}(v)]\sim\frac{e^{\gamma_E}}{v}+O(v^{-2}). \tag{F48}$$

We now analyze the behavior of the terms involving $F_R^{1\mathrm{noneq}}$ in Eq. (F39). Using the large v behavior of $F_R^{1\mathrm{noneq}}(v,u)$ (F30), one has

$$2\frac{v}{u-1}\int_0^1 ds\left\{F_R^0\left(\frac{u-s}{u-1}v\right)F_R^{1\mathrm{noneq}}\left(\frac{1-s}{u-1}v,\frac{1}{s}\right)+F_R^0\left(\frac{1-s}{u-1}v\right)F_R^{1\mathrm{noneq}}\left(\frac{u-s}{u-1}v,\frac{u}{s}\right)\right\}$$

$$\sim\frac{v}{u-1}\int_0^1 ds\left\{e^{-[(u-s)/(u-1)v]}\frac{(u-1)^2}{v^2(1-s)^2}\frac{(\sqrt{1/s}-1)^2}{\sqrt{1/s}}+e^{-[(1-s)/(u-1)v]}\frac{(u-1)^2}{v^2(u-s)^2}\frac{(\sqrt{u/s}-1)^2}{\sqrt{u/s}}\right\}. \tag{F49}$$

Notice first on this expression that we are left with convergent integrals over s. Moreover, in the large v limit, due to the exponential prefactors the first term decays also exponentially (for $u>1$), and the second one is dominated by $s=1$, which leads to a power law decay

$$\sim\frac{1}{v(u-1)}\frac{(\sqrt{u}-1)^2}{\sqrt{u}}\int_0^1 ds e^{-[(1-s)/(u-1)v]}\sim O(v^{-2}).$$

We are now left with the double integral in Eq. (F39), which is dominated—also due to the exponential prefactor—by $t_1\sim vu/(u-1)$ and $t_2\sim v/(u-1)$. Therefore to get the leading behavior, we substitute t_1 and t_2 by these values in the integrand (except, of course, in the exponential $e^{t_1+t_2}$). This yields

$$\frac{1}{2}e^{\gamma_E}e^{-v(1+u)/(u-1)}\int_0^{vu/(u-1)}dt_1\times\int_0^{v/(u-1)}dt_2\frac{\sqrt{t_1}-\sqrt{t_2}e^{t_1+t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}$$

$$=\frac{1}{2}e^{\gamma_E}\frac{(\sqrt{u}-1)^2}{\sqrt{u}}\frac{1}{v}e^{-v(u+1)/(u-1)}\times\int_0^{vu/(u-1)}dt_1e^{t_1}\int_0^{v/(u-1)}dt_2e^{t_2}+O\left(\frac{1}{v^2}\right)$$

$$=\frac{1}{2}e^{\gamma_E}\frac{(\sqrt{u}-1)^2}{\sqrt{u}}\frac{1}{v}+O\left(\frac{1}{v^2}\right), \tag{F50}$$

which together with the other term in v^{-1} (F48) yields Eq. (134) in the text.

c. The limit of vanishing momentum

To obtain the limit of vanishing momentum $\tilde{q}\to 0$ of the correlation function, we look at the behavior of $F_C(v,u)$ when $v\to 0$, up to order $O(v)$ terms [due to the q^{-2} prefactor in Eq. (F39)]. This is done in the following way:

$$2e^{\gamma_E}e^{-v(1+u)/(u-1)}\left[\mathrm{Ei}\left(\frac{2v}{u-1}\right)-\ln\left(\frac{2v}{u-1}\right)-\gamma_E\right]$$
$$+\frac{1}{2}e^{\gamma_E}\left\{e^{-v(u+1)/u-1}\left[-4-2\frac{uv}{u-1}\mathrm{Ei}\left(\frac{uv}{u-1}\right)\right.\right.$$
$$\left.-2\frac{v}{u-1}\mathrm{Ei}\left(\frac{v}{u-1}\right)\right]+2(e^{-vu/(u-1)}+e^{-v/(u-1)})$$
$$\left.+2[e^{-v}-1+ve^{-v}Ei(v)]\right\}$$
$$=\frac{e^{\gamma_E}v}{u-1}6-2\ln ve^{\gamma_E}-u\ln u+(u+1)\ln(u-1)+O(v^2). \quad \text{(F51)}$$

Then using the expression of $F_R^{1\text{noneq}}(0,u)$ (F33), one has

$$2\frac{v}{u-1}\int_0^1 ds\left\{F_R^0\left(\frac{u-s}{u-1}v\right)F_R^{1\text{noneq}}\left(\frac{1-s}{u-1}v,\frac{1}{s}\right)\right.$$
$$\left.+F_R^0\left(\frac{1-s}{u-1}v\right)F_R^{1\text{noneq}}\left(\frac{u-s}{u-1}v,\frac{u}{s}\right)\right\}$$
$$=\frac{2v}{u-1}\int_0^1 ds\left\{F_R^{1\text{noneq}}\left(0,\frac{1}{s}\right)+F_R^{1\text{noneq}}\left(0,\frac{u}{s}\right)\right\}+O(v^2)$$
$$=4e^{\gamma_E}\frac{v}{u-1}\int_0^1 \ln\frac{1+\sqrt{s}}{2}+\ln\frac{1+\sqrt{\frac{s}{u}}}{2}+O(v^2)$$
$$=\frac{4e^{\gamma_E}v}{u-1}\left[\sqrt{u}-(u-1)\ln\left(1+\frac{1}{\sqrt{u}}\right)-2\ln 4\right]+O(v^2). \quad \text{(F52)}$$

To treat the double integral in the last line of Eq. (F39), we come back to the variables $\tilde{t},\tilde{t}',\tilde{q}$,

$$\frac{1}{2}\frac{T}{\tilde{q}^2}\tau e^{\gamma_E}e^{-v(1+u)/(u-1)}\int_0^{vu/(u-1)}dt_1$$
$$\times\int_0^{v/(u-1)}dt_2\frac{(\sqrt{t_1}-\sqrt{t_2})e^{t_1+t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}$$
$$=\frac{1}{2}T\tau e^{\gamma_E}e^{-\tilde{q}^2(\tilde{t}+\tilde{t}')}\int_0^{\tilde{t}}dt_1\int_0^{\tilde{t}'}dt_2\frac{(\sqrt{t_1}-\sqrt{t_2})e^{q^2(t_1+t_2)}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}. \quad \text{(F53)}$$

Under this form, the limit $\tilde{q}\to$ is very simply obtained,

$$\lim_{\tilde{q}\to 0}\frac{1}{2}\frac{T}{\tilde{q}^2}\tau e^{\gamma_E}e^{-v(1+u)/(u-1)}\int_0^{vu/(u-1)}dt_1$$
$$\times\int_0^{v/(u-1)}dt_2\frac{(\sqrt{t_1}-\sqrt{t_2})e^{t_1+t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}$$
$$=\frac{1}{2}T\tau e^{\gamma_E}\int_0^{\tilde{t}}dt_1\int_0^{\tilde{t}'}dt_2\frac{\sqrt{t_1}-\sqrt{t_2}}{\sqrt{t_1t_2}(\sqrt{t_1}+\sqrt{t_2})}$$
$$=2T\tau e^{\gamma_E}\tilde{t}'\left((u-1)\ln(1+\sqrt{u})-\frac{u}{2}\ln u\right). \quad \text{(F54)}$$

Finally, Eqs. (F51), (F52), and (F54), together with Eq. (F39) and the complete expression of the correlation function $\mathcal{C}^{\tilde{q}}_{\tilde{t}\tilde{t}'}$, (131), lead to

$$\mathcal{C}^{\tilde{q}=0}_{\tilde{t}\tilde{t}'}=2T_c\tilde{t}'\left[1+\tau-\frac{z-2}{2}[\ln(\tilde{t}-\tilde{t}')+\gamma_E]\right.$$
$$\left.+\theta\ln\frac{\tilde{t}}{\tilde{t}'}+\tau F_C^{\text{diff1}}\left(\frac{\tilde{t}}{\tilde{t}'}\right)\right],$$
$$F_C^{\text{diff1}}(u)=\frac{1}{2}e^{\gamma_E}[4\sqrt{u}+(u+1)\ln(u-1)$$
$$-2(u-1)\ln(1+\sqrt{u})-2\ln u+6-8\ln 4], \quad \text{(F55)}$$

where we have used $F_C^0(v,u)=v/(u-1)+O(v^2)$, $\partial_v F_C^0(v,u)=1/(u-1)+O(v)$ and $v/(u-1)=\tilde{q}^2\tilde{t}'$: this gives the scaling form (135) given in the text.

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[62] The higher replica operators used here are defined in terms of excluded replica sums. They thus form a different basis than the one used in, e.g., FRG studies of zero temperature fixed point in higher dimension [17–19]. These are defined in terms of unrestricted sums directly related to the cumulant of the disorder, which is not the case here.